

A linear set view on KM-arcs

Maarten De Boeck *

Geertrui Van de Voorde †

Abstract

In this paper, we study KM-arcs of type t , i.e. point sets of size $q+t$ in $\text{PG}(2, q)$ such that every line contains 0, 2 or t of its points. We use field reduction to give a different point of view on the class of *translation* arcs. Starting from a particular \mathbb{F}_2 -linear set, called an *i-club*, we reconstruct the *projective triads*, the *translation hyperovals* as well as the translation arcs constructed by Korchmáros-Mazzocca, Gács-Weiner and Limbupasiriporn. We show the KM-arcs of type $q/4$ recently constructed by Vandendriessche are translation arcs and fit in this family.

Finally, we construct a family of KM-arcs of type $q/4$. We show that this family, apart from new examples that are not translation KM-arcs, contains all translation KM-arcs of type $q/4$.

Keywords: KM-arc, $(0, 2, t)$ -arc, set of even type, translation arc

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1 Introduction

1.1 KM-arcs

Point sets in $\text{PG}(2, q)$, the Desarguesian projective plane over the finite field \mathbb{F}_q of order q , that have few different intersections sizes with lines have been a research subject throughout the last decades. A point set \mathcal{S} of *type* (i_1, \dots, i_m) in $\text{PG}(2, q)$ is a point set such that for every line in $\text{PG}(2, q)$ the intersection size $\ell \cap \mathcal{S}$ equals i_j for some j and such that each value i_j occurs as intersection size for some line. In [11] point sets of type $(0, 2, q/2)$ of size $\frac{3q}{2}$ were studied. This led to the following generalisation by Korchmáros and Mazzocca in [5].

Definition 1.1. A *KM-arc of type t* in $\text{PG}(2, q)$ is a point set of type $(0, 2, t)$ with size $q + t$. A line containing i of its points is called an *i-secant*.

*This author is supported by the BOF-UGent (Special Research Fund of Ghent University).
UGent, Department of Mathematics, Krijgslaan 281 – S22, 9000 Gent, Flanders, Belgium.
Email: mdeboeck@cage.ugent.be

†This author is a postdoctoral fellow of the Research Foundation Flanders (FWO – Vlaanderen).
UGent, Department of Mathematics, Krijgslaan 281 – S22, 9000 Gent, Flanders, Belgium.
Email: gvdvoorde@cage.ugent.be

Originally these KM-arcs were denoted as $(q+t)$ -arcs of type $(0, 2, t)$ [5] or ‘ $(q+t, t)$ -arcs of type $(0, 2, t)$ ’ [2] but in honour of Korchmáros and Mazzocca we denote them by KM-arcs. The following results were obtained in [2, Theorem 2.5] and [5, Proposition 2.1].

Theorem 1.2. *If \mathcal{A} is a KM-arc of type t in $\text{PG}(2, q)$, $2 < t < q$, then*

- q is even;
- t is a divisor of q ;
- there are $\frac{q}{t} + 1$ different t -secants to \mathcal{A} , and they are concurrent.

If \mathcal{A} is a KM-arc of type t , then the point contained in all t -secants to \mathcal{A} is called the t -nucleus of \mathcal{A} .

Definition 1.3. A point set \mathcal{S} in $\text{PG}(2, q)$ is called a *translation set* with respect to the line ℓ if the group of elations with axis ℓ fixing \mathcal{S} acts transitively on the points of $\mathcal{S} \setminus \ell$; the line ℓ is called the *translation line*. If a KM-arc is a translation set, then it is called a *translation KM-arc*.

Theorem 1.4 ([5, Proposition 6.2]). *If $\mathcal{S} \subset \text{PG}(2, q)$ is a translation KM-arc of type t with respect to the line ℓ , then ℓ is a t -secant to \mathcal{S} .*

The main questions in the study of the KM-arcs are the following: *for which values of q and t does a KM-arc of type t in $\text{PG}(2, q)$ exist?* and *which nonequivalent KM-arcs of type t in $\text{PG}(2, q)$ exist for given admissible q and t ?* We give a survey of the known results in Table 1.

q	t	Condition	Comments	Reference
2^h	2^i	$h - i \mid h$	see Sections 3.1, 3.2 generalising [5, Sect. 5]	[2, Constr. 3.4(1))]
2^h	2^{i+1}	$h - i \mid h$	see Section 3.2	[2, Constr. 3.4(2))]
2^h	2^{i+m}	$h - i \mid h$, a KM-arc of type 2^m in $\text{PG}(2, 2^{h-i})$ exists	see Section 3.2	[2, Constr. 3.4(3))]
2^h	2^{h-2}	$h \geq 3$	translation KM-arcs	[12, Sect. 5] Sections 2.4 and 3.4
2^h	2^{h-2}	$h \geq 3$	new construction	Section 4
32	4		see Section 3.3	[4]
32	8		see Section 3.3	[10]
64	8		see Section 3.3	[10]

Table 1: An overview of the known KM-arcs

In this article we will use linear sets to study these problems. We will describe a new family of KM-arcs of type $q/4$, and look at known KM-arcs from this point of view.

It was noted a few years ago that KM-arcs together with their t -nucleus determine \mathbb{F}_2 -linear sets on each of their t -secants, however they are not \mathbb{F}_2 -linear sets themselves. Vandendriessche conjectured in a lecture [13] that this is always the case.

In the second half of this section we recall the basic information on field reduction and linear sets. In Section 2 we discuss translation KM-arcs. We introduce i -clubs and discuss their relationship with KM-arcs. We will prove a characterisation theorem of translation KM-arcs using these i -clubs, discuss KM-arcs of type $q/2$ and translation hyperovals from this perspective and describe a family of type $q/4$ using this setting. In Section 3 we will discuss the known KM-arcs from the point of view of linear sets and show that the family of Vandendriessche is a family of translation KM-arcs which can be constructed as in Section 2. In Section 4 we present a new family of KM-arcs of type $q/4$, including the family of Vandendriessche as well as many examples of non-translation arcs. We end by showing that every translation KM-arc of type $q/4$ is a member of this new family.

1.2 Linear sets and field reduction

A $(t-1)$ -spread \mathcal{S} of $\text{PG}(n-1, q)$ is a partition of the point set of $\text{PG}(n-1, q)$ into subspaces of dimension $(t-1)$. It is a classic result of Segre that a $(t-1)$ -spread of $\text{PG}(n-1, q)$ can only exist if t divides n . The construction of a *Desarguesian spread* that follows shows the well-known fact that this condition is also sufficient.

A Desarguesian $(t-1)$ -spread of $\text{PG}(rt-1, q)$ can be obtained by applying *field reduction* to the points of $\text{PG}(r-1, q^t)$. The underlying vector space of the projective space $\text{PG}(r-1, q^t)$ is $V(r, q^t)$; if we consider $V(r, q^t)$ as a vector space over \mathbb{F}_q , then it has dimension rt , so it defines a $\text{PG}(rt-1, q)$. In this way, every point P of $\text{PG}(r-1, q^t)$ corresponds to a subspace of $\text{PG}(rt-1, q)$ of dimension $(t-1)$ and it is not hard to see that this set of $(t-1)$ -spaces forms a spread of $\text{PG}(rt-1, q)$, which is called a Desarguesian spread. If U is a subset of $\text{PG}(rt-1, q)$, and \mathcal{D} a Desarguesian $(t-1)$ -spread, then we define $\mathcal{B}(U) := \{R \in \mathcal{D} \mid U \cap R \neq \emptyset\}$. In this paper, we consider the Desarguesian spread \mathcal{D} as fixed and we identify the elements of $\mathcal{B}(U)$ with their corresponding points of $\text{PG}(r-1, q^t)$.

Linear sets can be defined in several equivalent ways, but using the terminology introduced here, an \mathbb{F}_q -linear set S of rank h in $\text{PG}(r-1, q^t)$ is a set of points such that $S = \mathcal{B}(\mu)$, where μ is an $(h-1)$ -dimensional subspace of $\text{PG}(rt-1, q)$. The *weight* of a point $P = \mathcal{B}(p)$ of a linear set $\mathcal{B}(\mu)$ equals $\dim(\mu \cap p) + 1$. Following [7], a *club of rank h* is a linear set \mathcal{S} of rank h such that one point of \mathcal{S} has weight $h-1$ and all others have weight 1. We define an *i -club of rank h* as a linear set \mathcal{C} of rank h such that one point, called the *head* of \mathcal{C} has weight i and all others have weight 1. With this definition, a usual club of rank h is an $(h-1)$ -club. A 1-club is a *scattered linear set*, which is defined to be a linear set of which all points have weight one. We see that the head of a 1-club is not defined. Note that an \mathbb{F}_q -linear i -club of rank h in $\text{PG}(1, q^t)$ has size $q^{h-1} + q^{h-2} + \dots + q^i + 1$. For more information on field reduction and linear sets, we refer to [8].

2 Translation KM-arcs and i -clubs

2.1 A geometric construction for translation arcs

Let \mathcal{D} be the Desarguesian $(h-1)$ -spread in $\text{PG}(3h-1, 2)$ corresponding to $\text{PG}(2, 2^h)$, let ℓ_∞ be the line at infinity of $\text{PG}(2, 2^h)$ and let H be the $(2h-1)$ -space such that $\ell_\infty = \mathcal{B}(H)$. The points of $\text{PG}(2, 2^h)$ that are not on ℓ_∞ are the *affine* points.

Theorem 2.1. *Let μ be an $(h-1)$ -space in H such that $\mathcal{B}(\mu)$ is an i -club \mathcal{C} of rank h with head N in ℓ_∞ , and let $\rho \in \mathcal{D}$ be the spread element such that $\mathcal{B}(\rho) = N$. Let π be an h -space meeting H exactly in μ . Then the point set $\mathcal{B}(\pi) \setminus \mathcal{C}$ together with the points of $\ell_\infty \setminus \mathcal{C}$ forms a translation KM-arc of type 2^i in $\text{PG}(2, 2^h)$ with axis ℓ_∞ and with 2^i -nucleus N .*

Proof. We denote $(\mathcal{B}(\pi) \setminus \mathcal{C}) \cup (\ell_\infty \setminus \mathcal{C})$ by \mathcal{A} . As π is an h -space that meets H , which is spanned by spread elements, in an $(h-1)$ -space, a spread element that meets $\pi \setminus \mu$ non-trivially, meets it in a point. Consequently, \mathcal{A} has 2^h affine points. The size of $\mathcal{C} = \mathcal{B}(\mu)$ is $2^{h-1} + \dots + 2^i + 1$, which implies that \mathcal{A} contains $(2^h + 1) - (2^{h-1} + \dots + 2^i + 1) = 2^i$ points of ℓ_∞ . So in total, \mathcal{A} has $2^h + 2^i$ points.

Let ℓ be a line in $\text{PG}(2, 2^h)$ different from ℓ_∞ , and let L be the $(2h-1)$ -space in $\text{PG}(3h-1, 2)$ such that $\ell = \mathcal{B}(L)$. If $L \cap H = \rho$, then L contains the $(i-1)$ -space $\mu \cap \rho$. Since L contains no other points of H than the points of ρ , either $L \cap \pi$ is an i -space, or else $L \cap \pi$ equals the $(i-1)$ -space $\mu \cap \rho$. In the former case $|\mathcal{A} \cap \ell| = 2^i$, in the latter case ℓ contains no points of \mathcal{A} .

If $L \cap H$ is a spread element different from ρ , then L meets μ in a point or $L \cap \mu = \emptyset$. In the former case ℓ has no point in common with $\ell_\infty \setminus \mathcal{C}$, and L meets π in a line or a point, so $\ell \cap (\mathcal{B}(\pi) \setminus \mathcal{C})$ equals 0 or 2. In the latter case ℓ has one point in common with $\ell_\infty \setminus \mathcal{C}$, and L meets π in a point by the Grassmann identity, so $|\ell \cap (\mathcal{B}(\pi) \setminus \mathcal{C})|$ equals 1. Consequently, all lines meet \mathcal{A} in 0, 2 or 2^i points, and all lines that meet it in 2^i points, pass through N ; \mathcal{A} is a KM-arc of type 2^i with 2^i -nucleus N .

We now prove that \mathcal{A} is a translation KM-arc with axis ℓ_∞ . Let P_1 and P_2 be two points of $\mathcal{A} \setminus \ell_\infty$, and let $Q_1, Q_2 \in (\pi \setminus \mu)$ be the points such that $\mathcal{B}(Q_1) = P_1$ and $\mathcal{B}(Q_2) = P_2$. We consider the elation γ in the $(2h)$ -space $\langle H, \pi \rangle$ with axis H , that maps Q_1 on Q_2 . This elation induces an elation $\bar{\gamma}$ of $\text{PG}(2, 2^h)$ with axis $\mathcal{B}(H) = \ell_\infty$. Note that π is fixed by γ , and hence $\mathcal{B}(\pi)$ is fixed by $\bar{\gamma}$. So \mathcal{A} is fixed by $\bar{\gamma}$. Since $Q_1^\gamma = Q_2$, $P_1^{\bar{\gamma}} = P_2$. \square

Theorem 2.2. *Every translation KM-arc of type 2^i in $\text{PG}(2, 2^h)$ can be constructed as in Theorem 2.1.*

Proof. From [5, Proposition 6.3], we know that if \mathcal{A} is a translation KM-arc of type t in $\text{PG}(2, q)$, $q = 2^h$ with translation line $Z = 0$, and $(0, 1, 0)$ as t -nucleus, then the affine points of \mathcal{A} can be written as $(f(t), t, 1)$ where $f(z) = \sum_{i=0}^{h-1} \alpha_i z^{2^i}$ with $\alpha_i \in \mathbb{F}_2$.

Now let $\{\omega, \omega^2, \omega^{2^2}, \dots, \omega^{2^{h-1}}\}$ be a normal basis for \mathbb{F}_{2^h} over \mathbb{F}_2 and consider field reduction with respect to this basis, i.e. we let the vector (u, v, w) of $\mathbb{F}_{2^h}^3$ correspond to the vector $(u_0, \dots, u_{h-1}; v_0, \dots, v_{h-1}; w_0, \dots, w_{h-1})$ of \mathbb{F}_2^{3h} , where $u = \sum_{i=0}^{h-1} u_i \omega^{2^i}$, $v = \sum_{i=0}^{h-1} v_i \omega^{2^i}$ and $w = \sum_{i=0}^{h-1} w_i \omega^{2^i}$.

Write $1 = \sum_{i=0}^{h-1} a_i \omega^{2^i}$, and let $k \in \{0, \dots, h-1\}$ be an index for which $a_k = 1$. Let $t \in \mathbb{F}_{2h} = \sum_{i=0}^{h-1} t_i \omega^{2^i}$, then $t^{2^j} = \sum_{i=0}^{h-1} t_{h-j+i} \omega^{2^i}$, where the indices are taken modulo h . We see that $f(t) = \sum_{j=0}^{h-1} (\sum_{i=0}^{h-1} \alpha_j t_{h-i+j} \omega^{2^i})$, again with the indices taken modulo h .

This implies that every point $(f(t), t, 1)$, $t \in \mathbb{F}_{2h}$ is defined by a vector of $\mathbb{F}_{2^{3h}}$ corresponding to a point of $\text{PG}(3h-1, 2)$ that belongs to the h -dimensional subspace π defined by the $2h-1$ equations:

$$X_i = \sum_{j=i}^{h-1} \alpha_j X_{h-i+j} + \sum_{j=0}^{i-1} \alpha_j X_{2h-i+j}, i \in \{0, \dots, h-1\}$$

$$X_{2h+j} = a_j X_{2h+k}, j \in \{0, \dots, h-1\}, j \neq k.$$

The intersection of π with the $(2h-1)$ -space H corresponding to the line $z=0$, defined by $X_{2h} = X_{2h+1} = \dots = X_{3h-1} = 0$ satisfies one extra equation, namely $X_{2h+k} = 0$, hence, π meets H in an $(h-1)$ -dimensional space μ .

Since f is an \mathbb{F}_2 -linear map, the set of directions determined by the set $\{(f(t), t, 1) \mid t \in \mathbb{F}_{2h}\}$ equals $\{(f(z), z, 0) \mid z \in \mathbb{F}_{2h}\}$.

If \mathcal{A} is a KM-arc with affine part \mathcal{A}' then it is clear that the set of points of the KM-arc of type 2^i on the line at infinity is exactly the set of non-determined directions by \mathcal{A}' . The size of this set is 2^i , which shows that the set \mathcal{A}' determines $2^h - 2^i + 1$ directions and that $|\mathcal{B}(\mu)| = 2^h - 2^i + 1$. Since we know that the point $(0, 1, 0)$ lies on all 2^i -secants to the affine part of \mathcal{A} , determined by $\mathcal{B}(\pi) \setminus \mathcal{B}(\mu)$, we obtain that the spread element corresponding to $(0, 1, 0)$ meets μ in an $(i-1)$ -space. Consequently, all other spread elements that meet μ , meet it in a point, and $\mathcal{B}(\mu)$ is an i -club. \square

2.2 The case $i = h-1$ and projective triads

A *projective triad* \mathcal{T} of side n in $\text{PG}(2, q)$ is a set of $3n-2$ points, n on each of 3 concurrent lines ℓ_0, ℓ_1, ℓ_2 , with $\ell_0 \cap \ell_1 \cap \ell_2 = \{P\}$ such that a line through a point $R_0 \neq P$ of \mathcal{T} on ℓ_0 and $R_1 \neq P$ of \mathcal{T} on ℓ_1 meets ℓ_2 in a point R_2 of \mathcal{T} .

Theorem 2.3. *All \mathbb{F}_q -linear $(h-1)$ -clubs of rank h in $\text{PG}(1, q^h)$ are PGL-equivalent to the set $(x, \text{Tr}(x))_{\mathbb{F}_{q^h}}$, $x \in \mathbb{F}_{q^h}$, where Tr denotes the trace function from \mathbb{F}_{q^h} to \mathbb{F}_q .*

Proof. A set of points skew from $(0, 1)$, defined by $q^h - 1$ vectors, can be written as $(x, f(x))$, $x \in \mathbb{F}_{q^h}^*$ since all these points have a different first coordinate. This implies that every \mathbb{F}_q -linear set of rank h skew from $(0, 1)$ can be written as $\{(x, f(x)) \mid x \in \mathbb{F}_{q^h}^*\}$ where f is an \mathbb{F}_q -linear map. Clearly, every $(h-1)$ -club is PGL-equivalent to an $(h-1)$ -club \mathcal{S}' of rank h that has head $(1, 0)$ and is skew from $(0, 1)$. The spread element corresponding to $(1, 0)$ consists of all projective points corresponding to vectors of the form $(x, 0)$, $x \in \mathbb{F}_{q^h}$ and all hyperplanes of this h -dimensional subvector space are given by the points $(x, 0)$ where $\text{Tr}(\alpha x) = 0$ for some $\alpha \in \mathbb{F}_{q^h}^*$ (see [9, 2.24]). The element of PGL induced by the matrix $\begin{bmatrix} \alpha & 0 \\ 0 & 1 \end{bmatrix}$ maps the point set $(x, \text{Tr}(\alpha x))_{\mathbb{F}_{q^h}}$, $x \in \mathbb{F}_{q^h}^*$ onto $(x, \text{Tr}(x))_{\mathbb{F}_{q^h}}$, $x \in \mathbb{F}_{q^h}^*$ which proves the statement. \square

Lemma 2.4. *If μ and μ' are subspaces of dimension $h-1$ in $\text{PG}(2h-1, q)$ such that $\mathcal{B}(\mu) = \mathcal{B}(\mu')$ and such that $\mu \cap \mu'$ is a hyperplane of some spread element, then $\mu = \mu'$.*

Proof. Suppose to the contrary that $\mu \neq \mu'$ and let π be the h -space $\langle \mu, \mu' \rangle$. Since $\mathcal{B}(\mu) = \mathcal{B}(\mu')$, the q^{h-1} spread elements of $\mathcal{B}(\mu)$, different from $\mathcal{B}(\mu \cap \mu')$ meet π in a line. Moreover, by counting, we see that every point of π , not in $\mu \cap \mu'$ lies on such a spread element. But there are q^h spread elements different from $\mathcal{B}(\mu \cap \mu')$, each meeting π in a point or in a line, a contradiction. \square

Corollary 2.5. *There are $q^{\frac{q^h-1}{q-1}}$ different \mathbb{F}_q -linear $(h-1)$ -clubs of rank h with a fixed head in $\text{PG}(1, q^h)$, $h > 2$; there are $q(q^{2h}-1)/(q-1)$ different \mathbb{F}_q -linear $(h-1)$ -clubs of rank h in $\text{PG}(1, q^h)$, $h > 2$.*

Proof. Note that the restriction $h > 2$ is necessary, since for $h = 2$ an $(h-1)$ -clubs of rank h is a scattered set and the head of a scattered set is not defined.

There are (q^h+1) choices for the head of the $(h-1)$ -club. For a fixed head, we may fix a hyperplane μ of the spread element ρ corresponding to it since the elementwise stabiliser of the Desarguesian spread acts transitively on the hyperplanes of one spread element. By the previous lemma, we have that every $(h-1)$ -space through μ different from ρ gives rise to another $(h-1)$ -club with head $\mathcal{B}(\rho)$. We know that there are $\frac{q^{h+1}-1}{q-1} - 1 = q^{\frac{q^h-1}{q-1}}$ such $(h-1)$ -spaces. This gives in total $(q^h+1)(\frac{q^{h+1}-1}{q-1} - 1) = q^{\frac{q^{2h}-1}{q-1}}$ different \mathbb{F}_q -linear $(h-1)$ -clubs of rank h in $\text{PG}(1, q^h)$. \square

Corollary 2.6. *The stabiliser of an \mathbb{F}_q -linear $(h-1)$ -club in $\text{PG}(1, q^h)$, $h > 2$, in $\text{PTL}(2, q^h)$, $q = p^t$, p prime, has size $htq^{h-1}(q-1)$.*

We will characterise KM-arcs of type $q/2$ by showing that all projective triads are \mathbb{F}_2 -linear sets. For this, we need the following lemma by Vandendriessche.

Lemma 2.7 ([12, Corollary 4.4]). *On 3 fixed concurrent lines in $\text{PG}(2, q)$, q even, there are $4q-4$ projective triads.*

Lemma 2.8. *On 3 fixed concurrent lines of $\text{PG}(2, q)$, $q = 2^h$, there are $4q-4$ \mathbb{F}_2 -linear triads.*

Proof. Fix 3 concurrent lines, ℓ_1, ℓ_2, ℓ_3 , with $\ell_1 \cap \ell_2 = N$ and consider two points $R_1 \neq N \in \ell_1$ and $R_2 \neq N \in \ell_2$. The point $\langle R_1, R_2 \rangle \cap \ell_3$ is denoted by R_3 . Let S_i be the spread element in $\text{PG}(3h-1, 2)$ corresponding to R_i , $i = 1, 2, 3$, and let ρ be the spread element corresponding to N . If μ is an h -space determining (via the construction given in Theorem 2.1) an \mathbb{F}_2 -linear triad with nucleus N and containing R_1 and R_2 , then clearly, μ meets ρ in an $(h-2)$ -space and contains a transversal line to the regulus through S_1, S_2, S_3 . If $\mathcal{B}(\mu)$ is a triad through R_1 , we may always choose μ in such a way that it contains a fixed point Q of S_1 . If $\mathcal{B}(\mu)$ contains R_2 as well, we see that if μ contains Q , it contains the unique transversal line ℓ through Q to the regulus through S_1, S_2, S_3 . Now there are 2^h-1 hyperplanes of ρ , and each of these hyperplanes defines together with ℓ a different h -space. It is clear from Lemma 2.4 that the defined triads are also different. If we now count triples $(R_1 \neq N \text{ on } \ell_1, R_2 \neq N \text{ on } \ell_2, \text{ triad through } R_1 \text{ and } R_2)$, we find that $2^h \cdot 2^h \cdot (2^h-1) = X \cdot 2^{h-1} \cdot 2^{h-1}$, where X is the number of \mathbb{F}_2 -linear triads. \square

From the previous 2 lemmas, we immediately get:

Corollary 2.9. *All projective triads are \mathbb{F}_2 -linear sets.*

Theorem 2.10. *All projective triads of side $q/2$ in $\text{PG}(2, q)$, q even, are PGL -equivalent.*

Proof. Let (X, Y, Z) be coordinates for $\text{PG}(2, q)$, let \mathcal{T}_1 be a projective triad on the concurrent lines ℓ_1, ℓ_2, ℓ_3 , where $\ell_1 : X = 0$, $\ell_2 : Y = 0$, and $\ell_3 : X + Y = 0$ and let \mathcal{T}_2 be a projective triad on the concurrent lines m_1, m_2, m_3 . It is clear that there is a collineation φ mapping the lines m_1, m_2, m_3 onto ℓ_1, ℓ_2, ℓ_3 respectively. Let $\mathcal{T}_2' = \varphi(\mathcal{T}_2)$. From Lemma 2.3, we get that there is an element $\psi \in \text{PGL}(2, q)$ mapping $L_1 \cap \mathcal{T}_2'$ onto $L_1 \cap \mathcal{T}_1$, let $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ be a 2×2 -matrix corresponding to ψ . The 3×3 -matrix $\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ c & c & d \end{pmatrix}$ defines an element $\bar{\psi}$ of $\text{PGL}(3, q)$ that fixes the points on ℓ_2 and ℓ_3 , let $\mathcal{T}_2'' = \bar{\psi}(\mathcal{T}_2')$. Similarly, we can define an element ξ mapping $\mathcal{T}_2'' \cap \ell_2$ onto $\mathcal{T}_1 \cap \ell_2$ and fixing the points of ℓ_1 and ℓ_3 . Since a projective triad is uniquely defined by ℓ_3 and two sets of $\frac{q}{2}$ points on ℓ_1, ℓ_2 respectively, this implies that $\xi\psi\varphi(\mathcal{T}_2) = \mathcal{T}_1$, and hence \mathcal{T}_2 and \mathcal{T}_1 are PGL -equivalent. \square

Corollary 2.11. *All KM-arcs of type $q/2$ in $\text{PG}(2, q)$, q even, are PGL -equivalent to the example with affine points $\{(x, \text{Tr}(x), 1) \mid x \in \mathbb{F}_q\}$, where Tr denotes the absolute trace function $\mathbb{F}_q \rightarrow \mathbb{F}_2$.*

2.3 Translation hyperovals: $i = 1$

The construction of Theorem 2.1 also works for $i = 1$. In this case, we start with a 1-club in $\text{PG}(1, 2^h)$, i.e. a *scattered linear set*. The obtained KM-arc is an arc of type 2, which means that it is simply a *hyperoval*, and since it is a translation set, we obtain a *translation hyperoval*. The correspondence between translation hyperovals and scattered linear sets was already established in [3, Theorem 2].

2.4 A family of translation KM-arcs of type $q/4$ in $\text{PG}(2, q)$: $i = h - 2$

We construct an \mathbb{F}_q -linear $(h - 2)$ -club of rank h in $\text{PG}(1, q^h)$, and use Theorem 2.1 and this $(h - 2)$ -club to construct a family of KM-arcs of type $q/4$ in $\text{PG}(2, q)$, q even. Throughout this section we assume $h \geq 3$.

Lemma 2.12. *The set $\mathcal{C} =$*

$$\{(t_1\lambda + t_2\lambda^2 + \cdots + t_{h-1}\lambda^{h-1}, t_{h-1} + t_h\lambda) \mid t_i \in \mathbb{F}_q, (t_1, \dots, t_h) \neq (0, \dots, 0)\} \subseteq \text{PG}(1, q^h)$$

is an \mathbb{F}_q -linear $(h - 2)$ -club of rank h with head $(1, 0)$. Consequently, $|\mathcal{C}| = q^{h-1} + q^{h-2} + 1$.

Proof. The point set \mathcal{C} is determined by the \mathbb{F}_q -vector set

$$W = \{(0, t_1, t_2, \dots, t_{h-1}; t_{h-1}, t_h, 0, \dots, 0) \mid t_i \in \mathbb{F}_q, (t_1, \dots, t_h) \neq (0, \dots, 0)\} \subseteq \mathbb{F}_q^{2h}$$

It is immediate that \mathcal{C} is an \mathbb{F}_q -linear set of rank h , as $W \cup \{(0, \dots, 0)\}$ is an h -dimensional subspace of \mathbb{F}_q^{2h} .

The projective point $(1, 0)$ in \mathcal{C} arises from every vector in W with $t_{h-1} = t_h = 0$. Together with the zero vector, these form an $(h - 2)$ -dimensional subspace in \mathbb{F}_q^{2h} . All other projective points in \mathcal{C} are represented by $q - 1$ or by $q^2 - 1$ vectors in W . Assume

that a point is represented $q^2 - 1$ times, then we can find $t_1, \dots, t_{h-2}, t'_1, \dots, t'_{h-2} \in \mathbb{F}_q$ such that

$$(t_1\lambda + t_2\lambda^2 + \dots + t_{h-2}\lambda^{h-2} + \lambda^{h-1}, 1) = k(t'_1\lambda + t'_2\lambda^2 + \dots + t'_{h-2}\lambda^{h-2}, \lambda),$$

with $k \in \mathbb{F}_{q^h}$. Looking at the second coordinate we can see that $k = 1/\lambda$. However, $t_1\lambda + t_2\lambda^2 + \dots + t_{h-2}\lambda^{h-2} + \lambda^{h-1}$ and $t'_1 + t'_2\lambda + \dots + t'_{h-2}\lambda^{h-2}$ cannot be the same element of \mathbb{F}_{q^h} . Hence, all points in \mathcal{C} but $(1, 0)$ are represented by precisely $q - 1$ vectors in W , hence by precisely 1 point in the projective space $\text{PG}(2h - 1, q)$ arising from \mathbb{F}_q^{2h} . It follows that \mathcal{C} is an $(h - 2)$ -club. \square

The existence of an $(h - 2)$ -club of rank h in $\text{PG}(1, 2^h)$ established in Lemma 2.12, together with Theorem 2.1, yields the following result.

Theorem 2.13. *For every $h \geq 3$, there exists a translation KM-arc of type 2^{h-2} in $\text{PG}(2, 2^h)$. In other words, for even $q \geq 8$, there exists a translation set \mathcal{A} of size $q + q/4$ such that every line meets \mathcal{A} in 0, 2 or $q/4$ points.*

As mentioned before, it has been conjectured by Vandendriessche that the points of a KM-arc on a t -secant, together with the nucleus, form a linear set. Let \mathcal{A} be the KM-arc constructed above, with $q/4$ -nucleus $N = \mathcal{B}(\rho)$. By the construction described in Theorem 2.1 we know that $\{N\} \cup (\mathcal{A} \cap m)$ is a linear set for all four $q/4$ -secants m different from ℓ_∞ . In the following lemma, we check that the conjecture also holds for the line at infinity, which is the fifth $q/4$ -secant to \mathcal{A} . The lemma will be of use later in Section 3.4, when we will show that this family is equivalent to the family given by Vandendriessche.

Lemma 2.14. *Let \mathcal{C} be the $(h - 2)$ -club in $\ell = \text{PG}(1, q)$, $q = 2^h$, given in Lemma 2.12. Then the set $(\ell \setminus \mathcal{C}) \cup \{(1, 0)\}$ is an \mathbb{F}_2 -linear set.*

Proof. Let $f(x) = x^h + c_{h-1}x^{h-1} + \dots + c_1x + c_0$ be the primitive polynomial of the primitive element $\lambda \in \mathbb{F}_q$ that we used in the description of \mathcal{C} , so $f(\lambda) = 0$. We look at the set

$$\mathcal{C}' = \{(v_0 + v_1\lambda + v_2\lambda^2 + \dots + v_{h-3}\lambda^{h-3} + (v_0 + c_{h-1}v_{h-1})\lambda^{h-2} + v_{h-1}\lambda^{h-1}, v_0) \mid v_i \in \mathbb{F}_2, (v_0, \dots, v_{h-1}) \neq (0, \dots, 0)\} \subseteq \ell.$$

This is clearly an \mathbb{F}_2 -linear set of rank $h - 1$ which contains the projective point $(1, 0)$. All other projective points in \mathcal{C}' are the points in the above notation with $v_0 = 1$. Now we prove that $\mathcal{C} \cap \mathcal{C}' = \{(1, 0)\}$. Consider a point in the above set with $v_0 = 1$:

$$(1 + v_1\lambda + v_2\lambda^2 + \dots + v_{h-3}\lambda^{h-3} + (1 + c_{h-1}v_{h-1})\lambda^{h-2} + v_{h-1}\lambda^{h-1}, 1).$$

The \mathbb{F}_q -scalar multiples of this vector give rise to the same projective point. The only scalar multiples that we need to look at are

$$\begin{aligned} & (c_0v_{h-1} + (1 + c_1v_{h-1})\lambda + (v_1 + c_2v_{h-1})\lambda^2 + \dots + (v_{h-3} + c_{h-2}v_{h-1})\lambda^{h-2} + \lambda^{h-1}, \lambda) \text{ and} \\ & (1 + c_0v_{h-1} + (v_1 + 1 + c_1v_{h-1})\lambda + (v_2 + v_1 + c_2v_{h-1})\lambda^2 + \dots \\ & \quad + (v_{h-3} + v_{h-4} + c_{h-3}v_{h-1})\lambda^{h-3} + (1 + v_{h-3} + (c_{h-2} + c_{h-1})v_{h-1})\lambda^{h-2} \\ & \quad + (v_{h-1} + 1)\lambda^{h-1}, \lambda + 1) . \end{aligned}$$

Using field reduction with respect to the basis $\{1, \lambda, \dots, \lambda^{h-1}\}$ of \mathbb{F}_{2^h} over \mathbb{F}_2 , these correspond to the vectors

$$\begin{aligned} & (1, v_1, v_2, \dots, v_{h-3}, 1 + c_{h-1}v_{h-1}, v_{h-1}; 1, 0, \dots, 0), \\ & (c_0v_{h-1}, 1 + c_1v_{h-1}, v_1 + c_2v_{h-1}, \dots, v_{h-3} + c_{h-2}v_{h-1}, 1; 0, 1, 0, \dots, 0) \text{ and} \\ & (1 + c_0v_{h-1}, v_1 + 1 + c_1v_{h-1}, v_2 + v_1 + c_2v_{h-1}, \dots, \\ & \quad v_{h-3} + v_{h-4} + c_{h-3}v_{h-1}, 1 + v_{h-3} + (c_{h-2} + c_{h-1})v_{h-1}, v_{h-1} + 1; 1, 1, 0, \dots, 0). \end{aligned}$$

We can observe that none of these belong to the vector set W from Lemma 2.12. The lemma follows. \square

3 The known KM-arcs revisited

3.1 The arcs of Korchmáros and Mazzocca

Korchmáros and Mazzocca constructed a family of KM-arcs of type 2^i in $\text{PG}(2, 2^h)$, where $h - i \mid h$. Let L denote the relative trace function from \mathbb{F}_{2^h} to $\mathbb{F}_{2^{h-i}}$ and let g be an α -polynomial on $\mathbb{F}_{2^{h-i}}$. Then the authors show that the point set $\{(g(L(x)), x, 1) \mid x \in \mathbb{F}_{2^h}\}$ is the affine part of a KM-arc of type 2^i [5]. This family contains a subfamily of translation arcs; more precisely, the authors show the following.

Lemma 3.1 ([5, Proposition 6.4]). *The KM-arc with affine points $\{(g(L(x)), x, 1) \mid x \in \mathbb{F}_{2^h}\}$, where L denotes the relative trace function from \mathbb{F}_{2^h} to $\mathbb{F}_{2^{h-i}}$ is a translation arc of type 2^i in $\text{PG}(2, 2^h)$ if and only if $g(x) = x^{2^n}$ with $\gcd(n, h - i) = 1$.*

Let \mathcal{A} be a translation KM-arc of type 2^i in $\text{PG}(2, 2^h)$, obtained from the set

$$\{(L(x)^{2^n}, x, 1) \mid x \in \mathbb{F}_{2^h}\},$$

where $\gcd(h - i, n) = 1$ and L is the relative trace from \mathbb{F}_{2^h} to $\mathbb{F}_{2^{h-i}}$. By Theorem 2.2, we know that the affine points of \mathcal{A} determine an i -club \mathcal{C} on the line at infinity. This yields the following.

Theorem 3.2. *The translation KM-arcs of type 2^i in $\text{PG}(2, 2^h)$ constructed by Korchmáros and Mazzocca can be obtained from the construction of Theorem 2.1 by using the i -club $\{((L(x)^{2^n}, x) \mid x \in \mathbb{F}_{2^h})\}$ on the line at infinity, where L denotes the relative trace from \mathbb{F}_{2^h} to $\mathbb{F}_{2^{h-i}}$ and $\gcd(n, h - i) = 1$.*

As the existence and construction of \mathbb{F}_q -linear i -clubs in $\text{PG}(1, q^h)$ seems a non-trivial problem, we include a construction extending the example of an i -club obtained via the construction of Korchmáros and Mazzocca to an \mathbb{F}_q -linear i -club in $\text{PG}(1, q^h)$.

Theorem 3.3. *Let $h - i \mid h$, $i \geq 1$. The point set $\mathcal{C} = \{(L(x)^{q^n}, x) \mid x \in \mathbb{F}_{q^h}\}$, where L denotes the relative trace from \mathbb{F}_{q^h} to $\mathbb{F}_{q^{h-i}}$ and $\gcd(h - i, n) = 1$ defines an \mathbb{F}_q -linear i -club of rank h in $\text{PG}(1, q^h)$.*

Proof. Since $L(ax + y)^{q^n} = aL(x)^{q^n} + L(y)^{q^n}$ for $a \in \mathbb{F}_q$ and $x, y \in \mathbb{F}_{q^h}$, \mathcal{C} defines an \mathbb{F}_q -linear set which is clearly of rank h . The point $(0, 1)$ is defined by all non-zero vectors $(0, t)$ with $L(t)^{q^n} = 0$, and there are $(q^i - 1)$ such vectors. Moreover, if for some $x, y \in \mathbb{F}_{q^h}$ with $L(y) \neq 0$ the vectors $(L(x)^{q^n}, x)$ and $(L(y)^{q^n}, y)$ define the same projective point, then, by using that $L(x)^{q^n}/L(y)^{q^n} \in \mathbb{F}_{q^{h-i}}$, we obtain that $y = \lambda x$ for some $\lambda \in \mathbb{F}_{q^{h-i}}$. But this in turn yields that $\lambda^{q^n} = \lambda$. Since $\gcd(h - i, n) = 1$, this implies that $\lambda \in \mathbb{F}_q$, and hence, that \mathcal{C} is an \mathbb{F}_q -linear i -club in $\text{PG}(1, q^h)$. \square

3.2 The arcs of Gács and Weiner

In the paper [2], Gács and Weiner give a geometric construction of the KM-arcs of type 2^i in $\text{PG}(2, 2^h)$, $h - i \mid h$ described by Korchmáros and Mazzocca (see Section 3.1). Furthermore, they were able to extend this idea to a construction of KM-arc of type 2^{i+1} in $\text{PG}(2, 2^h)$, $h - i \mid h$. Finally, they gave a recursive construction, starting from a KM-arc of type 2^j in $\text{PG}(2, 2^{h-i})$, for a KM-arc of type 2^{i+j} in $\text{PG}(2, 2^h)$, $h - i \mid h$. It is worth mentioning that the authors end their paper with an algebraic description of the three families.

We now state their geometric construction in a different setting using field reduction.

Lemma 3.4. *Let \mathcal{D}' be the Desarguesian $(s - 1)$ -spread in $\text{PG}(3s - 1, 2^r)$ obtained by applying field reduction to the points of $\text{PG}(2, (2^r)^s)$. Let π be a plane of $\text{PG}(3s - 1, 2^r)$ such that $\mathcal{B}(\pi)$ spans the plane $\text{PG}(2, 2^{rs})$. Let P be a point of π , and let $\rho \in \mathcal{D}'$ be the spread element containing P . Let μ be a hyperplane of ρ , not through P . Let H be a hyperoval in π and let K be the cone with vertex μ and base H , i.e. $K = \bigcup_{Q \in H} \langle Q, \mu \rangle$. Then $\mathcal{B}(K) \setminus \mathcal{B}(\rho)$ is a KM-arc of type 2^{rs-r} in $\text{PG}(2, 2^{rs})$ if $P \in H$ and of type 2^{rs-r+1} if $P \notin H$.*

Moreover, if H' is a KM-arc of type 2^j in π with 2^j -nucleus P , then $\mathcal{B}(K') = \bigcup_{Q \in H'} \langle Q, \mu \rangle \setminus \mathcal{B}(\rho)$ is a KM-arc of type 2^{rs-r+j} in $\text{PG}(2, 2^{rs})$.

Proof. Let \mathcal{A} be the set $\mathcal{B}(K) \setminus \mathcal{B}(\rho)$ and $\mathcal{A}' = \mathcal{B}(K') \setminus \mathcal{B}(\rho)$. A line L of $\text{PG}(2, 2^{rs})$ corresponds to an $(2s - 1)$ -dimensional space ℓ in $\text{PG}(3s - 1, 2^r)$, spanned by spread elements. First suppose that $\rho \subset \ell$, then either ℓ meets π in the point P or in a line through the point P . In the first case, L is skew from \mathcal{A} and \mathcal{A}' , in the second, ℓ meets K outside ρ either in one or two $(s - 1)$ -spaces through μ , depending on whether or not P is a point of the hyperoval. This implies that in this case, L meets \mathcal{A} in 2^i or 2^{i+1} points, with $i = r(s - 1)$. As \mathcal{L} meets K' outside ρ in 2^j points, we see that L meets \mathcal{A}' in 2^{rs-r+j} points.

Now suppose that ℓ does not contain ρ . Let ξ be the $(s + 1)$ -space $\langle \pi, \rho \rangle$, then ℓ meets ξ in a line. Since K is a cone with base a hyperoval, a line skew from the vertex meets this cone in 0 or 2 points. Consequently, L contains 0 or 2 points of \mathcal{A} . Similarly, K' is a cone with base a KM-arc of type 2^j , hence, a line not meeting the subspace spanned by the vertex of the cone and the nucleus of the KM-arc meets K' in 0 or 2 points. \square

Using the algebraic description provided by the authors of [2], we see that if we use a translation hyperoval H in the construction of Lemma 3.4, we end up with a translation KM-arc, hence, we know that it corresponds to an i -club. When the point P is contained in H , we find the i -club of Theorem 3.2 but when the point P is not contained in H , we

can exploit the connection between i -clubs and translation KM-arcs to find an \mathbb{F}_2 -linear i -club in $\text{PG}(1, 2^h)$ where $h - i + 1 \mid h$.

Theorem 3.5. *The translation KM-arcs of type 2^i , $i = rt - r + 1$ in $\text{PG}(2, 2^h)$, $h = rt$ constructed by Gács and Weiner are PGL-equivalent to the ones obtained from Theorem 2.1 by using the i -club $\{x_0^{2^n} - ax_0, x_0 + \sum_{i=1}^{t-1} x_i \omega^i \mid x \in \mathbb{F}_{2^h}\}$ in $\text{PG}(1, 2^h)$, for some $a \in \mathbb{F}_{2^r}^*$ and $\gcd(r, n) = 1$ where $\{1, \omega, \dots, \omega^{t-1}\}$, $t > 1$ a basis for \mathbb{F}_{2^h} over \mathbb{F}_{2^r} .*

Also in this case, we can extend the construction of an i -club in $\text{PG}(1, 2^h)$ where $h - i + 1 \mid h$ to a construction of an i -club in $\text{PG}(1, q^h)$, $h - i + 1 \mid h$.

Theorem 3.6. *Let $f : \mathbb{F}_{q^r} \mapsto \mathbb{F}_{q^r}$, $r > 1$ be an \mathbb{F}_q -linear map such that $\{(x, f(x)) \mid x \in \mathbb{F}_{q^r}^*\}$ defines a scattered linear set in $\text{PG}(1, q^r)$. Consider the point set $\mathcal{C} = \{(f(x_0) - ax_0, bx_0 + \sum_{i=1}^{t-1} x_i \omega^i) \mid x_i \in \mathbb{F}_{q^r}, (x_0, \dots, x_{t-1}) \neq (0, \dots, 0)\}$ for some fixed $a, b \in \mathbb{F}_{q^r}$ and $\{1, \omega, \dots, \omega^{t-1}\}$, $t > 1$ a basis for $\mathbb{F}_{q^{rt}}$ over \mathbb{F}_{q^r} .*

- if $f(x) - ax = 0$ does not have non-zero solutions then \mathcal{C} defines an \mathbb{F}_q -linear $r(t-1)$ -club of rank rt in $\text{PG}(1, q^{rt})$.
- if $f(x) - ax = 0$ does have a non-zero solution, then \mathcal{C} defines an \mathbb{F}_q -linear $r(t-1)+1$ -club of rank rt in $\text{PG}(1, q^{rt})$.

Hence, there always exist \mathbb{F}_q -linear i -clubs of rank h in $\text{PG}(1, q^h)$ if $h - i \mid h$ and if $h - i + 1 \mid h$.

Proof. It is clear from the definition that \mathcal{C} is an \mathbb{F}_q -linear set of rank rt . If $f(x) - ax = 0$ does not have non-zero solutions, then there are $q^{r(t-1)} - 1$ vectors that determine the point $(0, 1)$.

If $\bar{x} \in \mathbb{F}_{q^r}$ is a non-zero solution of $f(x) - ax = 0$, then clearly $\lambda \bar{x}$ with $\lambda \in \mathbb{F}_q^*$ is another non-zero solution. If there would be a non-zero solution \bar{y} to $f(x) - ax = 0$, with \bar{y} not an \mathbb{F}_q -multiple of \bar{x} , then this would imply that the point $(1, a)$ contained in the scattered linear set $\{(x, f(x)) \mid x \in \mathbb{F}_{q^r}\}$ is defined by \bar{x} and \bar{y} which are not \mathbb{F}_q -multiples, a contradiction. The $q^{r(t-1)}$ values $b\bar{x} + \sum x_i \omega^i$ are different for every choice of \bar{x} , so we find in total $q^{r(t-1)+1} - 1$ different vectors determining the point $(0, 1)$.

Now we prove that all other points of \mathcal{C} are determined by a unique vector, up to \mathbb{F}_q -multiples. Suppose that $(\mu(f(x_0) - ax_0), \mu(bx_0 + \sum x_i \omega^i)) = (f(x'_0) - ax'_0, bx'_0 + \sum x'_i \omega^i)$, $\mu \in \mathbb{F}_{q^{rt}}$. We find that $\mu \sum x_i \omega^i = \frac{b}{a} f(x'_0) - \mu \frac{b}{a} f(x_0) + \sum x'_i \omega^i$. This implies that $x'_i = \mu x_i$ for $i \in \{1, \dots, t-1\}$ and that $f(x'_0) = \mu f(x_0)$. This in turn implies that $x'_0 = \mu x_0$. Finally, since f defines a scattered linear set, we see that $f(\mu x'_0) = \mu f(x_0)$ if and only if $\mu \in \mathbb{F}_q$.

We still need to prove the final statement. It is clear that we can always choose $a \in \mathbb{F}_{q^r}$ in such a way that $f(x) - ax = 0$ has a non-zero solution and since $|\{f(x)/x \mid x \in \mathbb{F}_{q^r}^*\}| < |\mathbb{F}_{q^r}|$, we can also choose an $a \in \mathbb{F}_{q^r}$ such that $f(x) - ax$ has no non-zero solution. Putting $h = rt$ and $i = rt - r + 1$ finishes the proof. \square

3.3 The arcs in $\text{PG}(2, 32)$ of Limbupasiriporn and Key, MacDonough and Mavron

In [10], Limbupasiriporn constructs KM-arcs of type 8 in $\text{PG}(2, 32)$. As these are all translation arcs, by Theorem 2.2 they can be obtained from the construction of Theorem

2.1. This implies that they correspond to a 3-club of rank 5 in $\text{PG}(1, 32)$. By computer, we checked that these 3-clubs are all PGL -equivalent, which implies the following.

Theorem 3.7. *All translation KM-arcs of type 8 in $\text{PG}(2, 32)$ are PGL -equivalent to the example from Theorem 2.13.*

Apart from the examples of Vandendriessche, which will be studied in Section 3.4, there is one example of a KM-arc in the literature that we did not cover yet, namely, the KM-arc of type 4 in $\text{PG}(2, 32)$ of Key, MacDonough and Mavron [4], which we now denote by \mathcal{A}_{kmm} . Note that the parameters $i = 2$ and $h = 5$ of \mathcal{A}_{kmm} do not fit in one of the infinite classes we have seen before. The arc \mathcal{A}_{kmm} is not a translation arc. We checked by computer that the 4 points on a 4-secant to \mathcal{A}_{kmm} , together with the 4-nucleus, do form an \mathbb{F}_2 -linear set.

3.4 The arcs of Vandendriessche

In [12] the author described a new family of KM-arcs of type $q/4$ in $\text{PG}(2, q)$. We start by presenting the construction of these examples.

Construction 3.8. Let λ be a primitive element of \mathbb{F}_q , $q = 2^h$, with minimal polynomial f , such that $f(x) = x^h + c_{h-3}x^{h-3} + \dots + c_1x + c_0$, with $c_i \in \mathbb{F}_2 \subset \mathbb{F}_q$ for all i . Note that $f(x)$ has no terms of degree $h-1$ and $h-2$. Each element $\mu \in \mathbb{F}_q$ can uniquely be written as $\mu_0 + \mu_1\lambda + \dots + \mu_{h-1}\lambda^{h-1}$ with $\mu_i \in \mathbb{F}_2 \subset \mathbb{F}_q$ for all $i = 0, \dots, h-1$.

Let $c \in \mathbb{F}_2 \subset \mathbb{F}_q$ be a parameter. We define the following sets:

$$\begin{aligned} \mathcal{S}_A &= \{(\mu, 1, 0) \mid \mu_{h-2} = 0, \mu_{h-3} = 1\} , \\ \mathcal{S}_B &= \{(\mu, 0, 1) \mid \mu_{h-1} = 0, \mu_{h-2} = 1\} , \\ \mathcal{S}_{C,c} &= \{(\mu, 1, 1) \mid \mu_{h-2} = 0, \sum_{i=0}^{h-3} \mu_i = c\} , \\ \mathcal{S}_{D,c} &= \{(\mu, \lambda, 1) \mid \mu_{h-1} + \mu_{h-2} = 1, \sum_{i=0}^{h-3} \mu_i = c\} , \\ \mathcal{S}_{E,c} &= \{(\mu, \lambda^2, 1) \mid \mu_{h-1} = 0, \sum_{i=0}^{h-2} \mu_i = c\} . \end{aligned}$$

Then $\mathcal{A}_c = \mathcal{S}_A \cup \mathcal{S}_B \cup \mathcal{S}_{C,c} \cup \mathcal{S}_{D,c} \cup \mathcal{S}_{E,c}$ is a KM-arc of type $q/4$.

Lemma 3.9. *The KM-arcs \mathcal{A}_0 and \mathcal{A}_1 from Construction 3.8 are PGL -equivalent.*

Proof. The collineation induced by the matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ takes \mathcal{A}_0 to \mathcal{A}_1 . □

The KM-arcs in $\text{PG}(2, q)$ described in [12] thus belong to one orbit of the collineation group $\text{PGL}(3, q)$.

Lemma 3.10. *The example in Construction 3.8 is a translation KM-arc.*

Proof. We consider the KM-arc \mathcal{A}_0 with the points given in Construction 3.8. Let ℓ be the $\frac{q}{4}$ -secant of \mathcal{A}_0 given by $\lambda Z = Y$. We prove that \mathcal{A}_0 is translation KM-arc with translation line ℓ . It is sufficient to check that for each point $P \in \mathcal{A}_0 \setminus \ell$ the elation φ_P mapping $(0, 1, 1) \in \mathcal{S}_{C,0}$ onto P , fixes \mathcal{A}_0 .

Assume P is a point $(\mu, 1, 0) \in \mathcal{S}_A$, then φ_P is given by

$$\begin{pmatrix} \lambda + 1 & \mu(\lambda + 1) & \mu(\lambda + 1)\lambda \\ 0 & 1 & \lambda^2 \\ 0 & 1 & 1 \end{pmatrix}.$$

This elation with center $((\lambda + 1)\mu, \lambda, 1)$ interchanges the lines $Z = 0$ and $Y = Z$, and the lines $Y = 0$ and $\lambda^2 Z = Y$. It can be calculated that each point of $\mathcal{A}_0 \setminus \ell$ is mapped onto another point of $\mathcal{A}_0 \setminus \ell$. E.g. the point $(x, 1, 0)$ with $x_{h-2} = 0$ and $x_{h-3} = 1$ is mapped onto the point $((\lambda + 1)(x + \mu), 1, 1)$, and

$$\begin{aligned} ((\lambda + 1)(x + \mu))_{h-2} &= x_{h-3} + x_{h-2} + \mu_{h-3} + \mu_{h-2} = 1 + 0 + 1 + 0 = 0 \\ \sum_{i=0}^{h-3} ((\lambda + 1)(x + \mu))_i &= x_{h-3} + \mu_{h-3} = 1 + 1 = 0, \end{aligned}$$

hence this is a point of $\mathcal{S}_{C,0}$.

If P is a point $(\mu, 0, 1) \in \mathcal{S}_B$, $(\mu, 1, 1) \in \mathcal{S}_{C,0}$, $(\mu, \lambda^2, 1) \in \mathcal{S}_{E,0}$, then φ_P is given by

$$\begin{aligned} &\begin{pmatrix} \lambda(\lambda + 1) & \mu(\lambda + 1) & \mu(\lambda + 1)\lambda \\ 0 & \lambda^2 & \lambda^2 \\ 0 & 1 & \lambda^2 \end{pmatrix}, \quad \begin{pmatrix} \lambda + 1 & \mu & \mu\lambda \\ 0 & \lambda + 1 & 0 \\ 0 & 0 & \lambda + 1 \end{pmatrix} \text{ and} \\ &\begin{pmatrix} \lambda(\lambda + 1) & \mu & \mu\lambda \\ 0 & 0 & \lambda^2(\lambda + 1) \\ 0 & \lambda + 1 & 0 \end{pmatrix}, \end{aligned}$$

respectively. In the calculations to show that each of the points of $\mathcal{A}_0 \setminus \ell$ is mapped onto a point of $\mathcal{A}_0 \setminus \ell$, the following equalities are useful:

$$\begin{aligned} \mathcal{S}_A &= \{(x, 1, 0) \mid x_{h-2} = 0, x_{h-3} = 1\} = \{(y, \lambda + 1, 0) \mid y_{h-2} = 1, \sum_{i=0}^{h-3} x_i = 1\} \\ &= \{(z, \lambda(\lambda + 1), 0) \mid z_{h-1} = 1, \sum_{i=0}^{h-2} z_i = 1\}, \\ \mathcal{S}_B &= \{(x, 0, 1) \mid x_{h-1} = 0, x_{h-2} = 1\} = \{(y, 0, \lambda + 1) \mid y_{h-1} = 1, \sum_{i=0}^{h-2} y_i = 1\}, \\ \mathcal{S}_{C,0} &= \{(x, 1, 1) \mid x_{h-2} = 0, \sum_{i=0}^{h-3} x_i = 0\} = \{(y, \lambda, \lambda) \mid y_{h-1} = 0, \sum_{i=0}^{h-2} y_i = 0\}. \end{aligned}$$

To obtain these equalities, we used the information on the primitive polynomial of λ . \square

Remark 3.11. By Theorem 2.2 we know that the points on the translation line ℓ not belonging to the KM-arc \mathcal{A}_0 , including the nucleus, determine an $(h-2)$ -club of rank h . So, the point set

$$\ell \setminus \mathcal{S}_{D,0} \cong \{(1,0)\} \cup \{(\mu,1) \mid \mu_{h-1} + \mu_{h-2} = 0 \vee \sum_{i=0}^{h-3} \mu_i = 1\}$$

is an $(h-2)$ -club of rank h .

Theorem 3.12. *The $(h-2)$ -club of rank h on the translation line of the KM-arc \mathcal{A}_0 of type $\frac{q}{4}$ given in Construction 3.8, consisting of the nucleus and the points not on the KM-arc, is PGL-equivalent to the $(h-2)$ -club \mathcal{C} of rank h given in Lemma 2.12.*

Proof. Let λ be a primitive element of \mathbb{F}_q , $q = 2^h$, with minimal polynomial f , such that $f(x) = x^h + c_{h-3}x^{h-3} + \dots + c_1x + c_0$, with $c_i \in \mathbb{F}_2 \subset \mathbb{F}_q$ for all i , as used in Construction 3.8.

The set of all points not in the $(h-2)$ -club of rank h corresponding to \mathcal{A}_0 is $\mathcal{S}_{D,0}$, and is equivalent to the set $\{(\mu,1) \mid \mu_{h-1} + \mu_{h-2} = 1, \sum_{i=0}^{h-3} \mu_i = 0\}$. By adding the point $(1,0)$ we find the linear set

$$\left\{ \left(\sum_{i=1}^{h-3} t_i + t_1\lambda + t_2\lambda^2 + \dots + t_{h-3}\lambda^{h-3} + t_{h-2}\lambda^{h-2} + (t_{h-2} + t_h)\lambda^{h-1}, t_h \right) \mid \right. \\ \left. t_i \in \mathbb{F}_2, (t_1, t_2, \dots, t_{h-2}, t_h) \neq (0, \dots, 0) \right\} ,$$

which is an $(h-2)$ -club of rank $h-1$ with head $(1,0)$. By looking at the collineation induced by $\begin{pmatrix} 1 & \lambda^{h-1} \\ 0 & 1 \end{pmatrix}$, we can see that this linear set is PGL-equivalent to the linear set

$$\mathcal{L} = \left\{ \left(\sum_{i=1}^{h-3} t_i + t_1\lambda + t_2\lambda^2 + \dots + t_{h-3}\lambda^{h-3} + t_{h-2}\lambda^{h-2} + t_{h-2}\lambda^{h-1}, t_h \right) \mid \right. \\ \left. t_i \in \mathbb{F}_2, (t_1, t_2, \dots, t_{h-2}, t_h) \neq (0, \dots, 0) \right\} .$$

By Lemma 2.14 and regarding the assumptions on the primitive polynomial f , the set $\mathcal{C}' = (\ell \setminus \mathcal{C}) \cup \{(1,0)\}$ is given by

$$\mathcal{C}' = \{(v_0 + v_1\lambda + v_2\lambda^2 + \dots + v_{h-3}\lambda^{h-3} + v_0\lambda^{h-2} + v_{h-1}\lambda^{h-1}, v_0) \mid \\ v_i \in \mathbb{F}_2, (v_0, \dots, v_{h-1}) \neq (0, \dots, 0)\} ,$$

which is also an $(h-2)$ -club of rank $h-1$ with head $(1,0)$. This set is PGL-equivalent to the linear set

$$\mathcal{L}' = \{(v_1\lambda + v_2\lambda^2 + \dots + v_{h-3}\lambda^{h-3} + v_{h-1}\lambda^{h-1}, v_0) \mid \\ v_i \in \mathbb{F}_2, (v_0, \dots, v_{h-1}) \neq (0, \dots, 0)\}$$

through the collineation $\begin{pmatrix} 1 & 1+\lambda^{h-2} \\ 0 & 1 \end{pmatrix}$. It is sufficient to prove that \mathcal{L} and \mathcal{L}' are PGL-equivalent to conclude that the $(h-2)$ -clubs $\ell \setminus \mathcal{S}_{D,0}$ and \mathcal{C} are PGL-equivalent. We note that the collineation induced by $\begin{pmatrix} 1+1/\lambda & 0 \\ 0 & 1 \end{pmatrix}$, maps \mathcal{L}' onto \mathcal{L} , which proves the theorem. \square

Remark 3.13. The translation KM-arcs that arise from the $(h-2)$ -club of rank h described in Section 2.4 are thus the KM-arcs that were previously described by Vandendriessche in [12]. However, the description presented in Lemma 2.12 needs no restrictions on the primitive polynomial of the underlying field \mathbb{F}_q of the projective plane $\text{PG}(2, q)$.

4 A new class of KM-arcs of type $\frac{q}{4}$

We now introduce two properties of KM-arcs of type $\frac{q}{4}$, that are satisfied by the KM-arc of Construction 3.8.

Definition 4.1. Let \mathcal{A} be a KM-arc of type $\frac{q}{4}$ in $\text{PG}(2, q)$, let N be its nucleus and let ℓ_0, \dots, ℓ_4 be its five $\frac{q}{4}$ -secants. Denote $\ell_i \cap \mathcal{A}$ by \mathcal{S}_i , with $i = 0, \dots, 4$, and denote

$$\mathcal{D}_{ij} = \{\langle P, Q \rangle \cap \ell_0 \mid P \in \mathcal{S}_i, Q \in \mathcal{S}_j\} \quad 1 \leq i < j \leq 4.$$

The KM-arc \mathcal{A} is said to have property (I) with respect to ℓ_0 if the following requirements are fulfilled.

1. The sets $\mathcal{S}_i \cup \{N\}$ are $(h-2)$ -clubs of rank $h-1$, for all $i = 1, \dots, 4$.
2. The following equalities are valid:

$$\mathcal{D}_{12} = \mathcal{D}_{34} \quad \mathcal{D}_{13} = \mathcal{D}_{24} \quad \mathcal{D}_{14} = \mathcal{D}_{23}.$$

3. $\mathcal{D}_{12} \cup \{N\}$, $\mathcal{D}_{13} \cup \{N\}$ and $\mathcal{D}_{14} \cup \{N\}$ are $(h-2)$ -clubs of rank $h-1$ with head N .
4. $\mathcal{D}_{12} \cap \mathcal{D}_{13} = \emptyset$, $\mathcal{D}_{12} \cap \mathcal{D}_{14} = \emptyset$ and $\mathcal{D}_{13} \cap \mathcal{D}_{14} = \emptyset$.

The KM-arc \mathcal{A} is said to have property (II) with respect to ℓ_0 if the following requirements are fulfilled.

1. The sets $\mathcal{S}_i \cup \{N\}$ are $(h-2)$ -clubs of rank $h-1$, for all $i = 1, \dots, 4$.
2. The following equalities are valid:

$$\mathcal{D}_{12} = \mathcal{D}_{34} \quad \mathcal{D}_{13} = \mathcal{D}_{24} \quad \mathcal{D}_{14} = \mathcal{D}_{23}.$$

3. $\mathcal{D}_{12} \cup \{N\}$, $\mathcal{D}_{13} \cup \{N\}$ and $\mathcal{D}_{14} \cup \{N\}$ are $(h-1)$ -clubs of rank h with head N .
4. $|\mathcal{D}_{12} \cap \mathcal{D}_{13}| = |\mathcal{D}_{12} \cap \mathcal{D}_{14}| = |\mathcal{D}_{13} \cap \mathcal{D}_{14}| = \frac{q}{4}$.
5. $\mathcal{D}_{12} \cap \mathcal{D}_{13} \cap \mathcal{D}_{14} = \emptyset$.

The first requirement in both properties in the previous definition asks that for any partition of the set $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4\}$ in two pairs, these two pairs of point sets determine the same points on ℓ_0 . From the other requirements it follows that $\ell_0 \setminus \{N\}$ is partitioned into four point sets of size $\frac{q}{4}$, namely \mathcal{S}_0 , $\mathcal{D}_{12} \cap \mathcal{D}_{13}$, $\mathcal{D}_{12} \cap \mathcal{D}_{14}$ and $\mathcal{D}_{13} \cap \mathcal{D}_{14}$ in case of property (I), and \mathcal{S}_0 , \mathcal{D}_{12} , \mathcal{D}_{13} and \mathcal{D}_{14} in case of property (II). So, the properties (I) and (II) look like two flavours of the same property.

Lemma 4.2. *A translation KM-arc of type $\frac{q}{4}$ in $\text{PG}(2, q)$, has property (I) with respect to its translation line.*

Proof. Let \mathcal{A} be a translation KM-arc of type $\frac{q}{4}$ in $\text{PG}(2, q)$, with ℓ its translation line. Denote its nucleus by N . Let \mathcal{D} be the Desarguesian $(h-1)$ -spread in $\text{PG}(3h-1, 2)$ corresponding to $\text{PG}(2, 2^h)$, let H be the $(2h-1)$ -space such that $\ell = \mathcal{B}(H)$, and let $\rho \in \mathcal{D}$ be the spread element such that $N = \mathcal{B}(\rho)$. By Theorem 2.2 we know that there is an $(h-1)$ -space $\mu \subset H$ and an h -space π meeting H exactly in μ such that $\mathcal{B}(\mu) = \ell \setminus \mathcal{A}$ and $\mathcal{B}(\pi) = \mathcal{A} \setminus \ell$. Moreover $\mu \cap \rho = \sigma$ is an $(h-3)$ -space, and μ meets all other spread elements in either a point or an empty space.

There are four $(h-2)$ -spaces through σ that are not contained in H . We denote them by σ_i , $i = 1, \dots, 4$. The point set $\mathcal{B}(\sigma_i) = \mathcal{A}_i \cup \{N\}$ is clearly an $(h-2)$ -club of rank $h-1$, $i = 1, \dots, 4$.

Using the notation from Definition 4.1 we see that $\mathcal{D}_{i,j}$ equals $\mathcal{B}(\langle \sigma_i, \sigma_j \rangle \cap H)$, $1 \leq i < j \leq 4$.

Consider the quotient geometry of σ in π ; this is a Fano plane \mathcal{F} in which $H \cap \pi$ is a line ℓ_H . The $(h-2)$ -spaces $\sigma_1, \dots, \sigma_4$ correspond to the points of \mathcal{F} not on ℓ_H . It is clear that \mathcal{D}_{ij} corresponds to the intersection of a line in \mathcal{F} with ℓ_H . This implies that $\mathcal{D}_{12} = \mathcal{D}_{34}$, $\mathcal{D}_{13} = \mathcal{D}_{24}$ and $\mathcal{D}_{14} = \mathcal{D}_{23}$. Moreover, \mathcal{D}_{12} , \mathcal{D}_{13} and \mathcal{D}_{14} correspond to the three $(h-2)$ -spaces through σ in μ . The two other requirements for property (I) follow immediately. \square

By Lemma 3.10 we know that the KM-arc of type $\frac{q}{4}$ given in Construction 3.8 is a translation KM-arc, so it has property (I) with respect to its translation line. We show that it has property (II) with respect to another $\frac{q}{4}$ -secant.

Lemma 4.3. *Let $\mathcal{A}_0 \subset \text{PG}(2, q)$ be the KM-arc of type $\frac{q}{4}$ described in Construction 3.8, let N be its nucleus, and let ℓ be the line $Z = 0$, which is a $\frac{q}{4}$ -secant of \mathcal{A} . Then \mathcal{A} has property (II) with respect to ℓ .*

Proof. Note that the line ℓ contains the point set \mathcal{S}_A . First, it is a direct observation that $\mathcal{S}_B \cup \{N\}$, $\mathcal{S}_{C,0} \cup \{N\}$, $\mathcal{S}_{D,0} \cup \{N\}$ and $\mathcal{S}_{E,0} \cup \{N\}$ are $(h-2)$ -clubs of rank $h-1$, given the form in which they are presented. Now, it is a straightforward calculation to see that

$$\begin{aligned} \{\langle P, Q \rangle \cap \ell \mid P \in \mathcal{S}_B, Q \in \mathcal{S}_{C,0}\} &= \{(x, 1, 0) \mid x_{h-2} = 1\} \text{ and} \\ \{\langle P, Q \rangle \cap \ell \mid P \in \mathcal{S}_{D,0}, Q \in \mathcal{S}_{E,0}\} &= \{(y, \lambda^2 + \lambda, 0) \mid \sum_{i=0}^{h-1} y_i = 1\}. \end{aligned}$$

Another calculation learns that these two point sets are equal. Hereby, we used the information on the primitive polynomial of λ given in Construction 3.8. We denote this point set by $\mathcal{D}_{BC|DE}$. Analogously we find that

$$\begin{aligned} \mathcal{D}_{BD|CE} &= \{(x, 1, 0) \mid x_{h-3} + x_{h-2} = 0\} = \{(y, \lambda, 0) \mid y_{h-2} + y_{h-1} = 0\} \\ &= \{(z, \lambda^2 + 1, 0) \mid \sum_{i=0}^{h-2} z_i = 0\}, \\ \mathcal{D}_{BE|CD} &= \{(x, 1, 0) \mid x_{h-3} = 0\} = \{(y, \lambda^2, 0) \mid y_{h-1} = 0\} = \{(z, \lambda + 1, 0) \mid \sum_{i=0}^{h-3} z_i = 0\}. \end{aligned}$$

It is immediate that $\mathcal{D}_{BC|DE} \cup \{N\}$, $\mathcal{D}_{BD|CE} \cup \{N\}$ and $\mathcal{D}_{BE|CD} \cup \{N\}$ are $(h-1)$ -clubs of rank h .

Now, we look at the equations $x_{h-2} = 1$, $x_{h-3} + x_{h-2} = 0$ and $x_{h-3} = 0$. Any two of them are independent, but no element of \mathbb{F}_q can satisfy all three. So, the intersections $\mathcal{D}_{BC|DE} \cap \mathcal{D}_{BD|CE}$, $\mathcal{D}_{BC|DE} \cap \mathcal{D}_{BE|CD}$ and $\mathcal{D}_{BD|CE} \cap \mathcal{D}_{BE|CD}$ have size $\frac{q}{4}$ and the intersection $\mathcal{D}_{BC|DE} \cap \mathcal{D}_{BD|CE} \cap \mathcal{D}_{BE|CD}$ is empty. It follows that \mathcal{A}_0 has property (II) with respect to ℓ . \square

Remark 4.4. The line ℓ is not necessarily the unique $\frac{q}{4}$ -secant such that \mathcal{A}_0 has property (II) with respect to it. E.g. for $q = 32$, the KM-arc \mathcal{A}_0 has property (II) with respect to each of the four $\frac{q}{4}$ -secants, different from the translation line.

We now present an easy to prove lemma on systems of equations with trace functions. We will need this lemma throughout the next theorems.

Lemma 4.5. *Let Tr be the absolute trace function $\mathbb{F}_q \rightarrow \mathbb{F}_2$, q even. Let k_1, \dots, k_m be m elements of \mathbb{F}_q and let c_1, \dots, c_m be elements in \mathbb{F}_2 . We consider the system of equations*

$$\begin{cases} \text{Tr}(k_1 x) = c_1 \\ \vdots \\ \text{Tr}(k_m x) = c_m \end{cases}.$$

Up to the ordering of the equations, we can assume that $\{k_1, \dots, k_{m'}\}$ is a basis of the \mathbb{F}_2 -subvector space of \mathbb{F}_q , generated by k_1, \dots, k_m . Let $a_{i,j} \in \mathbb{F}_2$ be such that $k_i = \sum_{j=1}^{m'} a_{i,j} k_j$ for all $i = m' + 1, \dots, m$ and $j = 1, \dots, m'$.

If $c_i = \sum_{j=1}^{m'} a_{i,j} c_j$ for all $i = m' + 1, \dots, m$, then this system has $\frac{q}{2^{m'}}$ solutions, otherwise it has no solutions.

In particular, the system has $\frac{q}{2^m}$ solutions if $\{k_1, \dots, k_m\}$ is an \mathbb{F}_2 -linear independent set.

We now describe a new family of KM-arcs of type $\frac{q}{4}$ that have property (I) or (II) with respect to one of its $\frac{q}{4}$ -secants.

Theorem 4.6. *Let Tr be the absolute trace function from \mathbb{F}_q to \mathbb{F}_2 , $q = 2^h$. Choose $\alpha, \beta \in \mathbb{F}_q \setminus \{0, 1\}$ such that $\alpha\beta \neq 1$ and define*

$$\gamma = \frac{\beta + 1}{\alpha\beta + 1}, \quad \xi = \alpha\beta\gamma.$$

Now choose $a, b \in \mathbb{F}_2 \subset \mathbb{F}_q$, and define the following sets

$$\begin{aligned} \mathcal{S}_0 &:= \left\{ (z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = 0, \text{Tr}\left(\frac{z}{\alpha}\right) = a \right\}, \\ \mathcal{S}_1 &:= \left\{ (z, 0, 1) \mid z \in \mathbb{F}_q, \text{Tr}(z) = 0, \text{Tr}\left(\frac{z}{\alpha\gamma}\right) = 0 \right\}, \\ \mathcal{S}_2 &:= \left\{ (z, 1, 1) \mid z \in \mathbb{F}_q, \text{Tr}(z) = 1, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = b \right\}, \\ \mathcal{S}_3 &:= \left\{ (z, \gamma, 1) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\alpha\gamma}\right) = a + 1, \text{Tr}\left(\frac{z}{\xi}\right) = b + 1 \right\}, \\ \mathcal{S}_4 &:= \left\{ (z, \beta + 1, 1) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + b + 1, \text{Tr}\left(\frac{z}{\xi}\right) = b \right\}. \end{aligned}$$

Then, $\mathcal{A} = \cup_{i=0}^4 \mathcal{S}_i$ is a KM-arc of type $\frac{q}{4}$ in $\text{PG}(2, q)$, and it has property (I) or (II) with respect to $Z = 0$.

Proof. First note that $\xi + \beta + \gamma = 1$. This observation is useful during the proof.

Clearly the lines $\ell_0 : Z = 0$, $\ell_1 : Y = 0$, $\ell_2 : Y = Z$, $\ell_3 : Y = \gamma Z$ and $\ell_4 : Y = (\beta + 1)Z$ each contain $\frac{q}{4}$ points of \mathcal{A} . For \mathcal{A} to be a KM-arc it is sufficient to check whether all other lines that contain at least two points of \mathcal{A} contain precisely two points of \mathcal{A} .

The line $\langle (x, 1, 0), (y, 0, 1) \rangle$ where $x, y \in \mathbb{F}_q$, with $\text{Tr}(x) = 0$, $\text{Tr}\left(\frac{x}{\alpha}\right) = a$, $\text{Tr}(y) = 0$ and $\text{Tr}\left(\frac{y}{\alpha\gamma}\right) = 0$ contains a point of both \mathcal{S}_0 and \mathcal{S}_1 , and contains the points $(x + y, 1, 1)$, $(x\gamma + y, \gamma, 1)$ and $(x(\beta + 1) + y, \beta + 1, 1)$ on ℓ_2, ℓ_3, ℓ_4 respectively. Now,

$$\begin{aligned} \text{Tr}(x + y) &= \text{Tr}(x) + \text{Tr}(y) = 0 + 0 = 0 \neq 1, \\ \text{Tr}\left(\frac{x\gamma + y}{\alpha\gamma}\right) &= \text{Tr}\left(\frac{x}{\alpha}\right) + \text{Tr}\left(\frac{y}{\alpha\gamma}\right) = a + 0 = a \neq a + 1, \\ \text{Tr}\left(\frac{x(\beta + 1) + y}{\alpha\beta}\right) + \text{Tr}\left(\frac{x(\beta + 1) + y}{\xi}\right) &= \text{Tr}\left(\frac{x}{\alpha}\right) + \text{Tr}\left(\frac{x\gamma}{\alpha\beta\gamma}\right) \\ &\quad + \text{Tr}\left(\frac{(1 + \beta)x}{\alpha\beta\gamma}\right) + \text{Tr}\left(\frac{y(\gamma + 1)}{\alpha\beta\gamma}\right) \\ &= a + \text{Tr}\left(\frac{(1 + \beta + \gamma)x}{\alpha\beta\gamma}\right) + \text{Tr}\left(\frac{y(\alpha + 1)}{\alpha\gamma(\alpha\beta + 1)}\right) \\ &= a + \text{Tr}(x) + \text{Tr}(y) + \text{Tr}\left(\frac{y}{\alpha\gamma}\right) \\ &= a + 0 + 0 + 0 = a \neq a + 1 = (a + b + 1) + b. \end{aligned}$$

So, $(x + y, 1, 1)$, $(x\gamma + y, \gamma, 1)$, $(x(\beta + 1) + y, \beta + 1, 1) \notin \mathcal{A}$. Analogously, on the line through $(x, 1, 0) \in \mathcal{S}_0$ and $(y, 1, 1) \in \mathcal{S}_2$ the points $(x(\gamma + 1) + y, \gamma, 1)$ and $(x\beta + y, \beta + 1, 1)$ are not contained in \mathcal{S}_3 and \mathcal{S}_4 , respectively, since

$$\begin{aligned} \text{Tr}\left(\frac{x(\gamma + 1) + y}{\alpha\gamma}\right) + \text{Tr}\left(\frac{x(\gamma + 1) + y}{\xi}\right) &= \text{Tr}\left(\frac{(x(\gamma + 1) + y)(\beta + 1)}{\alpha\beta\gamma}\right) \\ &= \text{Tr}\left(\frac{x}{\alpha}\right) + \text{Tr}\left(\frac{x(1 + \beta + \gamma)}{\alpha\beta\gamma}\right) \\ &\quad + \text{Tr}\left(\frac{y(\alpha\beta + 1)}{\alpha\beta}\right) \\ &= a + \text{Tr}(x) + \text{Tr}(y) + \text{Tr}\left(\frac{y}{\alpha\beta}\right) \\ &= a + b + 1 \neq a + b = (a + 1) + (b + 1), \\ \text{Tr}\left(\frac{x\beta + y}{\alpha\beta}\right) &= \text{Tr}\left(\frac{x}{\alpha}\right) + \text{Tr}\left(\frac{y}{\alpha\beta}\right) = a + b \neq a + b + 1. \end{aligned}$$

On the line through $(x, 1, 0) \in \mathcal{S}_0$ and $(y, \gamma, 1) \in \mathcal{S}_3$ the point $(x(\beta + \gamma + 1) + y, \beta + 1, 1)$ is not contained in \mathcal{S}_4 since

$$\text{Tr}\left(\frac{x(\beta + \gamma + 1) + y}{\xi}\right) = \text{Tr}\left(\frac{x(\beta + 1)\alpha\beta}{(\alpha\beta + 1)\alpha\beta\gamma}\right) + \text{Tr}\left(\frac{y}{\xi}\right) = \text{Tr}(x) + b + 1 = b + 1 \neq b.$$

On the line through $(x, 0, 1) \in \mathcal{S}_1$ and $(y, 1, 1) \in \mathcal{S}_2$ the points $(x(\gamma + 1) + y\gamma, \gamma, 1)$ and $(x\beta + y(\beta + 1), \beta + 1, 1)$ are not contained in \mathcal{S}_3 and \mathcal{S}_4 , respectively, since

$$\begin{aligned} \text{Tr} \left(\frac{x(\gamma + 1) + y\gamma}{\xi} \right) &= \text{Tr} \left(\frac{x(\alpha + 1)}{\alpha\gamma(\alpha\beta + 1)} \right) + \text{Tr} \left(\frac{y}{\alpha\beta} \right) = \text{Tr} \left(\frac{x(\alpha\gamma + 1)}{\alpha\gamma} \right) + b \\ &= \text{Tr}(x) + \text{Tr} \left(\frac{x}{\alpha\gamma} \right) + b = b \neq b + 1, \\ \text{Tr} \left(\frac{x\beta + y(\beta + 1)}{\xi} \right) &= \text{Tr} \left(\frac{x}{\alpha\gamma} \right) + \text{Tr} \left(\frac{y(\alpha\beta + 1)}{\alpha\beta} \right) = \text{Tr}(y) + \text{Tr} \left(\frac{y}{\alpha\beta} \right) = b + 1 \neq b. \end{aligned}$$

On the line through $(x, 0, 1) \in \mathcal{S}_1$ and $(y, \gamma, 1) \in \mathcal{S}_3$ the point $(x\frac{\beta+\gamma+1}{\gamma} + y\frac{\beta+1}{\gamma}, \beta + 1, 1)$ is not contained in \mathcal{S}_4 since

$$\begin{aligned} \text{Tr} \left(\frac{\frac{x(\beta+\gamma+1)}{\gamma} + \frac{y(\beta+1)}{\gamma}}{\alpha\beta} \right) &= \text{Tr} \left(\frac{x\xi}{\xi} \right) + \text{Tr} \left(\frac{y(\beta + 1)}{\alpha\beta\gamma} \right) = \text{Tr}(x) + \text{Tr} \left(\frac{y}{\alpha\beta} \right) + \text{Tr} \left(\frac{y}{\xi} \right) \\ &= a + b \neq a + b + 1. \end{aligned}$$

Finally, on the line through $(x, 1, 1) \in \mathcal{S}_2$ and $(y, \gamma, 1) \in \mathcal{S}_3$ the point $(x(1 + \frac{\beta}{\gamma+1}) + y\frac{\beta}{\gamma+1}, \beta + 1, 1)$ is not contained in \mathcal{S}_4 since

$$\begin{aligned} \text{Tr} \left(\frac{x(1 + \frac{\beta}{\gamma+1}) + y\frac{\beta}{\gamma+1}}{\alpha\beta} \right) + \text{Tr} \left(\frac{x(1 + \frac{\beta}{\gamma+1}) + y\frac{\beta}{\gamma+1}}{\xi} \right) &= \text{Tr} \left(\frac{x(1 + \beta + \gamma) + y\beta}{\xi} \right) \\ &= \text{Tr} \left(\frac{x(\beta + 1)}{\gamma(\alpha\beta + 1)} \right) + \text{Tr} \left(\frac{y}{\alpha\gamma} \right) \\ &= \text{Tr}(x) + a + 1 = a \neq a + 1. \end{aligned}$$

This finishes the proof that \mathcal{A} is a KM-arc of type $\frac{q}{4}$. We now show that it has property (I) or (II) with respect to the line ℓ_0 . The first requirement, which is the same for both property (I) and (II), is fulfilled: since it follows directly from the definition that $\mathcal{S}_i \cap \{N\}$ is an $(h - 2)$ -club of rank $h - 1$. Now, we need to distinguish between two cases.

First we assume that $\beta = \gamma$, which is equivalent to $\alpha\beta^2 = 1$. Using the notation from

Definition 4.1, we see that

$$\begin{aligned}
\mathcal{D}_{12} &= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = 1, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = b\} \text{ and} \\
\mathcal{D}_{34} &= \{(z, 1 + \beta + \gamma, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = b, \text{Tr}\left(\frac{z}{\xi}\right) = 1\} \\
&= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = b, \text{Tr}(z) = 1\}, \\
\mathcal{D}_{13} &= \{(z, \gamma, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = b + 1, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + 1\} \text{ and} \\
\mathcal{D}_{24} &= \{(z, \beta, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = b + 1, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + 1\}, \\
\mathcal{D}_{14} &= \{(z, \beta + 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = b, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + b + 1\} \text{ and} \\
\mathcal{D}_{23} &= \{(z, \gamma + 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = b, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + b + 1\}.
\end{aligned}$$

It follows that $\mathcal{D}_{12} = \mathcal{D}_{34}$, $\mathcal{D}_{13} = \mathcal{D}_{24}$ and $\mathcal{D}_{14} = \mathcal{D}_{23}$ since $\beta = \gamma$ and that $\mathcal{D}_{12} \cup \{N\}$, $\mathcal{D}_{13} \cup \{N\}$ and $\mathcal{D}_{14} \cup \{N\}$ are $(h - 2)$ -clubs of rank $h - 1$ with head N . So, the second and the third requirement of property (I) are fulfilled. Using $\alpha\beta^2 = 1$, we find that

$$\begin{aligned}
\mathcal{D}_{12} &= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = 1, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = b\} = \{(z, 1, 0) \mid \text{Tr}(z) = 1, \text{Tr}(\beta z) = b\} \\
\mathcal{D}_{13} &= \{(z, \beta, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = b + 1, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + 1\} \\
&= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(\beta z) = b + 1, \text{Tr}\left(\frac{z}{\alpha}\right) = a + 1\} \\
\mathcal{D}_{14} &= \{(z, \beta + 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = b, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + b + 1\} \\
&= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}((\beta + 1)z) = b, \text{Tr}(\beta(\beta + 1)z) = a + b + 1\} \\
&= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(\beta z) + \text{Tr}(z) = b, \text{Tr}(\beta z) + \text{Tr}\left(\frac{z}{\alpha}\right) = a + b + 1\}
\end{aligned}$$

and from this observation the final requirement of property (I), namely that $\mathcal{D}_{12} \cap \mathcal{D}_{13} = \mathcal{D}_{12} \cap \mathcal{D}_{14} = \mathcal{D}_{13} \cap \mathcal{D}_{14} = \emptyset$ follows. So, in this case \mathcal{A} has property (I) with respect to ℓ_0 .

Now, we assume that $\beta \neq \gamma$. We find

$$\begin{aligned}
\mathcal{D}_{12} &= \{(z, 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}(z) = 1\} \text{ and} \\
\mathcal{D}_{34} &= \{(z, 1 + \beta + \gamma, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{1 + \beta + \gamma}\right) = 1\} = \{(z, 1, 0) \mid \text{Tr}(z) = 1\}, \\
\mathcal{D}_{13} &= \{(z, \gamma, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\alpha\gamma}\right) = a + 1\} = \{(z, 1, 0) \mid \text{Tr}\left(\frac{z}{\alpha}\right) = a + 1\} \text{ and} \\
\mathcal{D}_{24} &= \{(z, \beta, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\alpha\beta}\right) = a + 1\} = \{(z, 1, 0) \mid \text{Tr}\left(\frac{z}{\alpha}\right) = a + 1\}, \\
\mathcal{D}_{14} &= \{(z, \beta + 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z(\alpha + 1)}{\alpha(\beta + 1)}\right) = a + 1\} \\
&= \{(z, 1, 0) \mid \text{Tr}\left(\frac{z(\alpha + 1)}{\alpha}\right) = a + 1\} \text{ and} \\
\mathcal{D}_{23} &= \{(z, \gamma + 1, 0) \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z(\alpha + 1)}{\alpha(\gamma + 1)}\right) = a + 1\} \\
&= \{(z, 1, 0) \mid \text{Tr}\left(\frac{z(\alpha + 1)}{\alpha}\right) = a + 1\}.
\end{aligned}$$

We observe that $\mathcal{D}_{12} = \mathcal{D}_{34}$, $\mathcal{D}_{13} = \mathcal{D}_{24}$ and $\mathcal{D}_{14} = \mathcal{D}_{23}$, so the second requirement of property (II) is fulfilled. It is clear that the sets $\mathcal{D}_{12} \cup \{N\}$, $\mathcal{D}_{13} \cup \{N\}$ and $\mathcal{D}_{14} \cup \{N\}$, with $N(1, 0, 0)$ the nucleus of \mathcal{A} , are $(h - 1)$ -clubs of rank h , so also the third requirement is fulfilled. By Lemma 4.5, the intersections $\mathcal{D}_{12} \cap \mathcal{D}_{13}$, $\mathcal{D}_{12} \cap \mathcal{D}_{14}$ and $\mathcal{D}_{13} \cap \mathcal{D}_{14}$ each contain $\frac{q}{4}$ points. The final requirement, that $\mathcal{D}_{12} \cap \mathcal{D}_{13} \cap \mathcal{D}_{14} = \emptyset$ follows from the observation that

$$z + \frac{z}{\alpha} = \frac{(\alpha + 1)z}{\alpha}. \quad \square$$

Remark 4.7. Recall that the construction of Theorem 4.6 depends on four parameters $\alpha, \beta \in \mathbb{F}_q \setminus \{0, 1\}$ and $a, b \in \mathbb{F}_2$, $\alpha\beta \neq 1$. Therefore, we denote this KM-arc of type $\frac{q}{4}$ by $\mathcal{A}_{\alpha, \beta, a, b}$.

Theorem 4.8. *Let $\alpha, \beta \in \mathbb{F}_q \setminus \{0, 1\}$, with $\alpha\beta \neq 1$, and $a, a', b, b' \in \mathbb{F}_2$. The KM-arcs $\mathcal{A}_{\alpha, \beta, a, b}$ and $\mathcal{A}_{\alpha, \beta, a', b'}$ are PGL-equivalent.*

Proof. Let $\nu \in \mathbb{F}_q$ be an element such that $\text{Tr}(\nu) = 0$ and $\text{Tr}(\frac{\nu}{\alpha}) = a + a'$. By Lemma 4.5 we know such an element exists since $\alpha \neq 1$. Denote $\text{Tr}(\frac{\nu}{\alpha\beta})$ by k .

Let γ be as in the construction of $\mathcal{A}_{\alpha, \beta, a, b}$ in Theorem 4.6. Now, let ρ be an element in \mathbb{F}_q such that $\text{Tr}(\rho) = 0$, $\text{Tr}(\frac{\rho}{\alpha\gamma}) = 0$ and $\text{Tr}(\frac{\rho}{\alpha\beta}) = k + b' + b$. Such an element exists by Lemma 4.5 since

$$1 + \frac{1}{\alpha\gamma} + \frac{1}{\alpha\beta} = \frac{\alpha\beta\gamma + \gamma + \beta}{\alpha\beta\gamma} = \frac{\alpha\beta\gamma + 1 + \alpha\beta\gamma}{\alpha\beta\gamma} = \frac{1}{\alpha\beta\gamma} \neq 0.$$

Now, the collineation induced by the matrix $\begin{pmatrix} 1 & \nu & \rho \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ maps $\mathcal{A}_{\alpha, \beta, a, b}$ on $\mathcal{A}_{\alpha, \beta, a', b'}$. To check that the sets \mathcal{S}_3 and \mathcal{S}_4 are indeed mapped onto points of $\mathcal{A}_{\alpha, \beta, a', b'}$ it is useful to note

that

$$\begin{aligned}\mathrm{Tr}\left(\frac{\rho}{\xi}\right) &= \mathrm{Tr}(\rho) + \left(\frac{\rho}{\alpha\beta}\right) + \mathrm{Tr}\left(\frac{\rho}{\alpha\gamma}\right) = k + b + b' \\ \mathrm{Tr}\left(\frac{\nu(\beta+1)}{\alpha\beta}\right) &= \mathrm{Tr}\left(\frac{\nu}{\alpha}\right) + \mathrm{Tr}\left(\frac{\nu}{\alpha\beta}\right) = k + a + a' \\ \mathrm{Tr}\left(\frac{\nu(\beta+1)}{\xi}\right) &= \mathrm{Tr}\left(\frac{\nu(\alpha\beta+1)}{\alpha\beta}\right) = k.\end{aligned}\quad \square$$

Theorem 4.9. *Let $\alpha, \beta \in \mathbb{F}_q \setminus \{0, 1\}$, with $\alpha\beta \neq 1$. The KM-arc $\mathcal{A}_{\alpha, \beta, 0, 0}$ is a translation KM-arc if and only if $\alpha \in \{\frac{1}{\beta^2}, 1 + \frac{1}{\beta}, \beta, \frac{1}{\sqrt{\beta}}, \frac{1}{\beta+1}\}$.*

Proof. First we prove that this condition on α and β is necessary. We denote the nucleus $(1, 0, 0)$ of $\mathcal{A}_{\alpha, \beta, 0, 0}$ by N . Assume that the $\frac{q}{4}$ -secant ℓ is a translation line of $\mathcal{A}_{\alpha, \beta, 0, 0}$. Denote the other four $\frac{q}{4}$ -secants by ℓ_1, \dots, ℓ_4 . Let $G \leq \mathrm{PGL}(3, q)$ be the group containing all elations of $\mathrm{PG}(2, q)$ with axis ℓ that fix $\mathcal{A}_{\alpha, \beta, 0, 0}$. We consider the natural action of G on the set \mathcal{L} of lines through N . The elation φ acts trivially on \mathcal{L} if and only if N is the center of φ . Let $H \leq G$ be the subgroup of elations with center N . If N is not the center of φ , then φ fixes precisely one element of \mathcal{L} , namely ℓ .

The set $\mathcal{L}' = \{\ell_1, \ell_2, \ell_3, \ell_4\} \subset \mathcal{L}$ is stabilised by the elements of G since G maps the $\frac{q}{4}$ -secants necessarily onto each other. We know that an element $\varphi \in G \setminus H$ cannot have order 3, for if $\varphi^3 = 1$, then the unique line in $\mathcal{L}' \setminus \{\ell_1, \ell_1^\varphi, \ell_1^{\varphi^2}\}$ has to be fixed, a contradiction. Consequently, φ^4 acts trivially on \mathcal{L} for all $\varphi \in G$.

Now, we assume that ℓ is the line $Z = 0$. Let $Y = x_i Z$ be the equation of the line ℓ_i . Any elation φ with axis ℓ is given by a matrix $A = \begin{pmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & w \end{pmatrix}$, with $u, v, w \in \mathbb{F}_q$. If $\varphi \in G$, then $w^4 = 1 \Leftrightarrow w = 1$ since $\ell_i^{\varphi^4} = \ell_i$ for all $i = 1, \dots, 4$. We can find a $\varphi \in G$ such that $\ell_1^\varphi = \ell_2$ since ℓ is a translation line. We find that $x_2 = x_1 + v$. Moreover, it also follows that $\ell_3^\varphi = \ell_4$ (also $\ell_2^\varphi = \ell_1$ and $\ell_4^\varphi = \ell_3$). So, $x_4 = x_3 + v$. Consequently, $x_1 + x_2 + x_3 + x_4 = 0$. Applying this to $\mathcal{A}_{\alpha, \beta, 0, 0}$, we find that if $Z = 0$ is a translation line then

$$0 + 1 + \gamma + (1 + \beta) = 0 \quad \Leftrightarrow \quad \beta = \gamma \quad \Leftrightarrow \quad \alpha = \frac{1}{\beta^2}.$$

Using a collineation in $\mathrm{PGL}(3, q)$ that stabilises N and that maps ℓ_i onto the line $Z = 0$, we can find the necessary conditions for the other $\frac{q}{4}$ -secants to be the translation line. The lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4 are translation lines if

$$\begin{aligned}0 + 1 + \frac{1}{\gamma} + \frac{1}{1 + \beta} &= 0 \quad \Leftrightarrow \quad \gamma = 1 + \frac{1}{\beta} & \Leftrightarrow \quad \alpha = 1 + \frac{1}{\beta} \\ 1 + 0 + \frac{\gamma}{\gamma + 1} + \frac{1 + \beta}{\beta} &= 0 \quad \Leftrightarrow \quad \gamma = \frac{1}{\beta + 1} & \Leftrightarrow \quad \alpha = \beta \\ \frac{1}{\gamma} + \frac{1}{\gamma + 1} + \frac{1}{\beta + \gamma + 1} + 0 &= 0 \quad \Leftrightarrow \quad \gamma = \sqrt{\beta + 1} & \Leftrightarrow \quad \alpha = \frac{1}{\sqrt{\beta}} \\ 0 + \frac{1}{\beta + 1} + \frac{1}{\beta} + \frac{1}{\beta + \gamma + 1} &= 0 \quad \Leftrightarrow \quad \gamma = \beta^2 + 1 & \Leftrightarrow \quad \alpha = \frac{1}{\beta + 1},\end{aligned}$$

respectively. This concludes the first part of the proof.

Now, we need to prove that the condition $\alpha \in \{\frac{1}{\beta^2}, 1 + \frac{1}{\beta}, \beta, \frac{1}{\sqrt{\beta}}, \frac{1}{\beta+1}\}$ is sufficient for $\mathcal{A}_{\alpha,\beta,0,0}$ to be a translation KM-arc. We show that $\ell : Z = 0$ is a translation line if $\alpha = \frac{1}{\beta^2}$. We consider the point $P(0, 0, 1) \in \mathcal{A}_{\alpha,\beta,0,0} \setminus \ell$. It is sufficient to prove that for any point $R \in \mathcal{A}_{\alpha,\beta,0,0} \setminus \ell$ the unique elation with axis ℓ mapping P onto R fixes $\mathcal{A}_{\alpha,\beta,0,0}$.

Any point in $\mathcal{A}_{\alpha,\beta,0,0} \setminus \ell$ can be written in one of the following ways:

$$(\mu_1, 0, 1), \text{ with } \text{Tr}(\mu_1) = 0, \text{Tr}(\beta\mu_1) = 0, \quad (\mu_2, 1, 1), \text{ with } \text{Tr}(\mu_2) = 1, \text{Tr}(\beta\mu_2) = 0, \\ (\mu_3, \beta, 1), \text{ with } \text{Tr}(\mu_3) = 1, \text{Tr}(\beta\mu_3) = 1, \quad (\mu_4, \beta + 1, 1), \text{ with } \text{Tr}(\mu_4) = 0, \text{Tr}(\beta\mu_4) = 1.$$

It can be calculated that the elations with axis ℓ given by the matrices

$$\begin{pmatrix} 1 & 0 & \mu_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \mu_2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \mu_3 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & \mu_4 \\ 0 & 1 & \beta + 1 \\ 0 & 0 & 1 \end{pmatrix}$$

fix the KM-arc $\mathcal{A}_{\alpha,\beta,0,0}$, and map P onto $(\mu_1, 0, 1)$, $(\mu_2, 1, 1)$, $(\mu_3, \beta, 1)$ and $(\mu_4, \beta + 1, 1)$, respectively. Consequently, $Z = 0$ is a translation line.

The arguments for the other $\frac{q}{4}$ -secants are analogous. \square

Remark 4.10. For a given $\beta \in \mathbb{F}_q \setminus \{0, 1\}$, q even, we consider the set $\Theta = \{\frac{1}{\beta^2}, 1 + \frac{1}{\beta}, \beta, \frac{1}{\sqrt{\beta}}, \frac{1}{\beta+1}\}$. If two of the elements in Θ coincide, then they all coincide, and Θ contains precisely one element. In this case, $\beta^3 = 1$. So, $\alpha = \beta$ is necessarily contained in the subfield $\mathbb{F}_4 \subseteq \mathbb{F}_q$, and hence q must be a square. So, if q is a square, and β is contained in $\mathbb{F}_4 \subseteq \mathbb{F}_q$, then there is only one value for α such that $\mathcal{A}_{\alpha,\beta,0,0}$ is a translation arc, but in this case it is translation with respect to all of its $\frac{q}{4}$ -secants. If q is a square, but β is not contained in $\mathbb{F}_4 \subseteq \mathbb{F}_q$, or if q is not a square, then there are precisely five values for α such that $\mathcal{A}_{\alpha,\beta,0,0}$ is a translation arc. In this case it is a translation arc with respect to only one of its $\frac{q}{4}$ -secants.

Remark 4.11. It is not hard to check that $\mathcal{A}_{\frac{1}{\beta^2},\beta,0,0} \cong \mathcal{A}_{\frac{1}{\beta^2+1},\beta+1,0,0}$, so by choosing different parameters α, β , we may end up with PGL-equivalent KM-arcs of the form $\mathcal{A}_{\alpha,\beta,0,0}$. However, as follows from Theorem 4.9, not all KM-arcs of the form $\mathcal{A}_{\alpha,\beta,0,0}$ are translation KM-arcs, so the family $\mathcal{A}_{\alpha,\beta,0,0}$ where α and β vary, certainly contains KM-arcs which are not PGL-equivalent.

In Theorems 4.12 and 4.13, we will prove that KM-arcs that satisfy property (I) or property (II) arise from the construction of Theorem 4.6. Before we give the proof, we introduce some notation to make a distinction between points of $\text{PG}(2, 2^h)$ and points of $\text{PG}(3h - 1, 2)$. We will denote a point of the latter space by a vector determining this point, using the subscript \mathbb{F}_2 . So for a fixed $x \in \mathbb{F}_{q^h}^*$, $(x, \text{Tr}(x), 0)_{\mathbb{F}_2}$ is a point of $\text{PG}(3h - 1, 2)$. Note that $\{(x, \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_{q^h}^*\}$ and $\{(\zeta x, \zeta \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_{q^h}^*\}$, with $\zeta \in \mathbb{F}_q \setminus \{0, 1\}$ are different projective subspaces of $\text{PG}(3h - 1, 2)$, but they determine the same (linear) point set in $\text{PG}(2, 2^h)$.

Theorem 4.12. *Let \mathcal{A} be a KM-arc of type $\frac{q}{4}$ in $\text{PG}(2, q)$, with $q = 2^h$. If ℓ is a $\frac{q}{4}$ -secant of \mathcal{A} such that \mathcal{A} has property (II) with respect to ℓ , then \mathcal{A} is PGL-equivalent to an arc given in Theorem 4.6 with $\alpha\beta^2 \neq 1$.*

Proof. Recall that Tr is the absolute trace function from \mathbb{F}_q to \mathbb{F}_2 . Let N be the nucleus of \mathcal{A} . Denote the four $\frac{q}{4}$ -secants of \mathcal{A} different from ℓ by $\ell_1, \ell_2, \ell_3, \ell_4$, and denote $\mathcal{A} \cap \ell_i$ by \mathcal{A}_i and $\mathcal{A}_i \cup \{N\}$ by \mathcal{A}_i^N , $i = 1, \dots, 4$. We also use the notations

$$\mathcal{D}_{ij} = \{\langle P, Q \rangle \cap \ell \mid P \in \mathcal{A}_i, Q \in \mathcal{A}_j\} \text{ and } \mathcal{D}_{ij}^N = \mathcal{D}_{ij} \cup \{N\} \quad 1 \leq i < j \leq 4.$$

Since \mathcal{A} has property (II) we know that $\mathcal{D}_{12} = \mathcal{D}_{34}$ and that \mathcal{D}_{12}^N is an $(h-1)$ -club of rank h with head N in ℓ . By Theorem 2.3 we can choose a frame of $\text{PG}(2, q)$ such that \mathcal{D}_{12}^N is the set $\{(x, \text{Tr}(x), 0) \mid x \in \mathbb{F}_q^*\}$, such that $(0, 0, 1) \in \mathcal{A}_1$ and $(1, 1, 1) \in \ell_2$. It follows that $(1, 0, 0)$ is the nucleus N of \mathcal{A} and that $\mathcal{D}_{12} = \{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1\}$. As all $(h-1)$ -clubs of rank h are PGL -equivalent and as \mathcal{D}_{13}^N has head N , there exists an $\alpha \in \mathbb{F}_q \setminus \{0, 1\}$ and an $\alpha' \in \mathbb{F}_q$ such that $\mathcal{D}_{13}^N = \{(\alpha x + \alpha' \text{Tr}(x), \text{Tr}(x), 0) \mid x \in \mathbb{F}_q^*\}$. Denote $\text{Tr}(\frac{\alpha'}{\alpha}) = a$. It follows that the set $\mathcal{D}_{13} = \mathcal{D}_{24}$ equals $\{(\alpha x + \alpha', 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1\}$. Now, \mathcal{D}_{14}^N is the unique $(h-1)$ -club of rank h with head N that has $\frac{q}{4}$ points in common with both \mathcal{D}_{12} and \mathcal{D}_{13} , but none with $\mathcal{D}_{12} \cap \mathcal{D}_{13}$. Hence, \mathcal{D}_{14}^N equals $\{(\frac{\alpha}{\alpha+1}x + \frac{\alpha'}{\alpha+1}\text{Tr}(x), \text{Tr}(x), 0) \mid x \in \mathbb{F}_q^*\}$ and consequently $\mathcal{D}_{14} = \mathcal{D}_{23}$ equals $\{(\frac{\alpha}{\alpha+1}x + \frac{\alpha'}{\alpha+1}, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1\}$.

Let \mathcal{D} be a Desarguesian spread in $\text{PG}(3h-1, 2)$, let H be the $(2h-1)$ -space such that $\ell = \mathcal{B}(H)$, and let $\rho \in \mathcal{D}$ be the spread element such that $\mathcal{B}(\rho) = N$.

We define σ as the $(2h)$ -space in $\text{PG}(3h-1, 2)$ spanned by H and $(0, 0, 1)_{\mathbb{F}_2}$. Then, there is a unique $(h-2)$ -space $\sigma_i \subset \sigma$ such that $\mathcal{B}(\sigma_i) = \mathcal{A}_i^N$, $i = 1, \dots, 4$. For every $i = 1, \dots, 4$, the subspace σ_i meets ρ in an $(h-3)$ -space; the existence of these subspaces σ_i follows from the first condition for KM-arcs of property (II). Let μ_{ij} be the subspace $H \cap \langle \sigma_i, \sigma_j \rangle$, for $1 \leq i < j \leq 4$. Then, $\mathcal{D}_{ij}^N = \mathcal{B}(\mu_{ij})$, for $1 \leq i < j \leq 4$.

The Desarguesian spread \mathcal{D} is fixed, and we have that $\mathcal{B}(\mu_{12}) = \{(x, \text{Tr}(x), 0) \mid x \in \mathbb{F}_q^*\}$. This leaves us the freedom to choose coordinates of $H = \text{PG}(2h-1, 2)$ in such a way that $\mu_{12} = \{(x, \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*\}$. Now we can find $\xi, \beta, \gamma, \delta, \varepsilon \in \mathbb{F}_q^*$ such that

$$\begin{aligned} \mu_{34} &= \{(\xi x, \xi \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*\} \\ \mu_{13} &= \{(\gamma(\alpha x + \alpha' \text{Tr}(x)), \gamma \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*\} \\ \mu_{24} &= \{(\beta(\alpha x + \alpha' \text{Tr}(x)), \beta \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*\} \\ \mu_{14} &= \left\{ \left(\delta \left(\frac{\alpha}{\alpha+1}x + \frac{\alpha'}{\alpha+1}\text{Tr}(x) \right), \delta \text{Tr}(x), 0 \right)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^* \right\} \\ \mu_{23} &= \left\{ \left(\varepsilon \left(\frac{\alpha}{\alpha+1}x + \frac{\alpha'}{\alpha+1}\text{Tr}(x) \right), \varepsilon \text{Tr}(x), 0 \right)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^* \right\}. \end{aligned}$$

The $(h-2)$ -space σ_1 must necessarily meet ρ in the $(h-3)$ -space $\tau_1 = \mu_{12} \cap \mu_{13} = \{(x, 0, 0) \mid x \in \mathbb{F}_q^*, \text{Tr}(x) = \text{Tr}(\frac{x}{\alpha\gamma}) = 0\}$. By Lemma 4.5 it follows that $\alpha\gamma \neq 1$. We know that μ_{14} must pass through $\mu_{12} \cap \mu_{13}$. Again by Lemma 4.5, we find the necessary condition that

$$1 + \frac{1}{\alpha\gamma} + \frac{\alpha+1}{\alpha\delta} = 0 \quad \Leftrightarrow \quad \delta = \frac{(\alpha+1)\gamma}{\alpha\gamma+1}.$$

Analogously, we know that $\tau_2 = \mu_{12} \cap \mu_{23} \cap \mu_{24}$, $\tau_3 = \mu_{13} \cap \mu_{23} \cap \mu_{34}$ and $\tau_4 = \mu_{14} \cap \mu_{24} \cap \mu_{34}$ are $(h-3)$ -spaces in ρ . It follows that $\alpha\beta \neq 1$, $\gamma(\alpha+1) \neq \varepsilon$ and $\beta(\alpha+1) \neq \delta$, and

moreover

$$\begin{aligned} 1 + \frac{1}{\alpha\beta} + \frac{\alpha+1}{\alpha\varepsilon} = 0 &\Leftrightarrow \varepsilon = \frac{(\alpha+1)\beta}{\alpha\beta+1}, \\ \frac{1}{\alpha\gamma} + \frac{\alpha+1}{\alpha\varepsilon} + \frac{1}{\xi} = 0 &\Leftrightarrow \frac{1}{\beta} + \frac{1}{\gamma} = \alpha \left(1 + \frac{1}{\xi}\right). \end{aligned}$$

Note that the equation $\frac{\alpha+1}{\alpha\delta} + \frac{1}{\alpha\beta} + \frac{1}{\xi} = 0$ derived from τ_4 is redundant.

If the $(h-3)$ -spaces τ_i and τ_j coincide, then $\langle\sigma_i, \sigma_j\rangle \cap H$ cannot be an $(h-1)$ -space. Hence, no two of the $(h-3)$ -spaces τ_1, \dots, τ_4 may coincide. Consequently, $\beta \neq \gamma$ and $\xi \neq 1$.

Since the set \mathcal{A}_1^N contains $(0, 0, 1)$ and since $\sigma_1 \subset \sigma$, the subspace σ_1 must contain $(0, 0, 1)_{\mathbb{F}_2}$. Hence, the points of $\sigma_1 \setminus H$ are the points $\{(x, 0, 1)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = \text{Tr}(\frac{x}{\alpha\gamma}) = 0\}$ and consequently $\mathcal{A}_1 = \{(x, 0, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = \text{Tr}(\frac{x}{\alpha\gamma}) = 0\}$.

Let $\rho' \in \mathcal{D}$ be the spread element such that $\mathcal{B}(\rho')$ is the point $(1, 1, 1)$. Then the intersection $\rho' \cap \sigma$ only contains one point, namely $(1, 1, 1)_{\mathbb{F}_2}$. The $(h-2)$ -space σ_2 is contained in $\langle\rho, \rho'\rangle$ since $(1, 1, 1) \in \ell_2$. Hence, it is contained in the h -space $\langle\rho, (1, 1, 1)_{\mathbb{F}_2}\rangle = \langle\rho, (0, 1, 1)_{\mathbb{F}_2}\rangle$. Furthermore, the $(h-2)$ -space σ_2 passes through the $(h-3)$ -space $\mu_{12} \cap \mu_{24} = \{(x, 0, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = \text{Tr}(\frac{x}{\alpha\beta}) = 0\} \subset \rho$. So, there is a $t' \in \mathbb{F}_q$ such that $\sigma_2 = \langle(t', 1, 1)_{\mathbb{F}_2}, \mu_{12} \cap \mu_{24}\rangle$. Note that $\text{Tr}(t') = 1$ since $\langle(t', 1, 1), (0, 0, 1)\rangle \cap \ell = (t', 1, 0)$ has to be a point of \mathcal{D}_{12} . Denote $\text{Tr}(\frac{t'}{\alpha\beta})$ by b , then, \mathcal{A}_2 is given by $\{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\frac{x}{\alpha\beta}) = b\}$.

The $(h-2)$ -space σ_3 passes through the $(h-3)$ -space $\mu_{13} \cap \mu_{34} = \{(x, 0, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(\frac{x}{\xi}) = \text{Tr}(\frac{x}{\alpha\gamma}) = 0\} \subset \rho$, and is contained in the h -spaces $\langle\mu_{13}, (0, 0, 1)_{\mathbb{F}_2}\rangle$ and $\langle\mu_{23}, (t', 1, 1)_{\mathbb{F}_2}\rangle$. So, the points of $\sigma_3 \setminus H$ are the points given by the following coordinates for both an x and a y in \mathbb{F}_q :

$$(\gamma\alpha x + \gamma\alpha'\text{Tr}(x), \gamma\text{Tr}(x), 1)_{\mathbb{F}_2} = \left(\frac{\varepsilon\alpha}{\alpha+1}y + \frac{\varepsilon\alpha'}{\alpha+1}\text{Tr}(y) + t', \varepsilon\text{Tr}(y) + 1, 1 \right)_{\mathbb{F}_2}.$$

Since \mathcal{A}_3 is not on the lines ℓ_1 or ℓ_2 , necessarily $\gamma\text{Tr}(x) = \varepsilon\text{Tr}(y) + 1 \notin \{0, 1\}$. Thus $\text{Tr}(x) = \text{Tr}(y) = 1$, and moreover also $\gamma = \varepsilon + 1$. Combining this with previous results $\varepsilon = \frac{(\alpha+1)\beta}{\alpha\beta+1}$, $\delta = \frac{(\alpha+1)\gamma}{\alpha\gamma+1}$ and $\frac{1}{\xi} = 1 + \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma}$, we find

$$\gamma = \frac{\beta+1}{\alpha\beta+1} \quad \text{and also} \quad \xi = \alpha\beta\gamma \quad \text{and} \quad \delta = \beta+1.$$

Now, as σ_3 passes through $\mu_{13} \cap \mu_{34}$ we know that the points of $\sigma_3 \setminus H$ are given by

$$\begin{aligned} &\left\{ (z, \gamma, 1)_{\mathbb{F}_2} \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\xi}\right) = d, \text{Tr}\left(\frac{z}{\alpha\gamma}\right) = d' \right\} \\ &= \left\{ (z, \gamma, 1)_{\mathbb{F}_2} \mid z \in \mathbb{F}_q, \text{Tr}\left(\frac{z}{\xi}\right) = d, \text{Tr}\left(\frac{z}{\xi}\right) + \text{Tr}\left(\frac{z}{\alpha\gamma}\right) = d + d' \right\} \end{aligned}$$

for some $d, d' \in \mathbb{F}_2$. We find that

$$\begin{aligned} d' &= \text{Tr} \left(\frac{\gamma\alpha x + \gamma\alpha' \text{Tr}(x)}{\alpha\gamma} \right) = \text{Tr}(x) + \text{Tr} \left(\frac{\alpha'}{\alpha} \right) = a + 1 \\ d + d' &= \text{Tr} \left(\frac{\alpha + 1}{\alpha\varepsilon} \left(\frac{\alpha\varepsilon y + \varepsilon\alpha'}{\alpha + 1} + t' \right) \right) = \text{Tr}(y) + \text{Tr} \left(\frac{\alpha'}{\alpha} \right) + \text{Tr}(t') + \text{Tr} \left(\frac{t'}{\alpha\beta} \right) \\ &= 1 + a + 1 + b = a + b. \end{aligned}$$

Hence, $d = b + 1$ and the points of \mathcal{A}_3 are given by $\{(x, \gamma, 1) \mid x \in \mathbb{F}_q, \text{Tr}(\frac{x}{\alpha\gamma}) = a + 1, \text{Tr}(\frac{x}{\xi}) = b + 1\}$.

The $(h-2)$ -space σ_4 passes through $\mu_{24} \cap \mu_{34} = \{(x, 0, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*, \text{Tr}(\frac{x}{\xi}) = \text{Tr}(\frac{x}{\alpha\beta}) = 0\} \subset \rho$ and is contained in the h -spaces $\langle \mu_{14}, (0, 0, 1)_{\mathbb{F}_2} \rangle$ and $\langle \mu_{24}, (t', 1, 1)_{\mathbb{F}_2} \rangle$. We proceed in the same way as for σ_3 . The points of $\sigma_4 \setminus H$ are the points given by the following coordinates for both an x and a y in \mathbb{F}_q :

$$\left(\frac{\delta\alpha}{\alpha + 1}x + \frac{\delta\alpha'}{\alpha + 1}\text{Tr}(x), (\beta + 1)\text{Tr}(x), 1 \right)_{\mathbb{F}_2} = (\beta\alpha y + \beta\alpha'\text{Tr}(y) + t', \beta\text{Tr}(y) + 1, 1)_{\mathbb{F}_2}.$$

Since $(\beta + 1)\text{Tr}(x) = \beta\text{Tr}(y) + 1 \notin \{0, 1\}$, necessarily $\text{Tr}(x) = \text{Tr}(y) = 1$. Now, as σ_4 passes through $\mu_{24} \cap \mu_{34}$ we know that the points of $\sigma_4 \setminus H$ are given by

$$\begin{aligned} &\left\{ (z, \beta + 1, 1)_{\mathbb{F}_2} \mid z \in \mathbb{F}_q, \text{Tr} \left(\frac{z}{\xi} \right) = e, \text{Tr} \left(\frac{z}{\alpha\beta} \right) = e' \right\} \\ &= \left\{ (z, \beta + 1, 1)_{\mathbb{F}_2} \mid z \in \mathbb{F}_q, \text{Tr} \left(\frac{z}{\alpha\beta} \right) = e', \text{Tr} \left(\frac{z}{\xi} \right) + \text{Tr} \left(\frac{z}{\alpha\beta} \right) = e + e' \right\} \end{aligned}$$

for some $e, e' \in \mathbb{F}_2$. We find that

$$\begin{aligned} e' &= \text{Tr} \left(\frac{\beta\alpha y + \beta\alpha'\text{Tr}(y) + t'}{\alpha\beta} \right) = \text{Tr}(y) + \text{Tr} \left(\frac{\alpha'}{\alpha} \right) + \text{Tr} \left(\frac{t'}{\alpha\beta} \right) = 1 + a + b \\ e + e' &= \text{Tr} \left(\frac{\alpha + 1}{\alpha\delta} \left(\frac{\alpha\delta x + \delta\alpha'}{\alpha + 1} \right) \right) = \text{Tr}(x) + \text{Tr} \left(\frac{\alpha'}{\alpha} \right) = 1 + a. \end{aligned}$$

We conclude that the points of \mathcal{A}_4 are given by $\{(x, \beta + 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(\frac{x}{\xi}) = b, \text{Tr}(\frac{x}{\alpha\beta}) = a + b + 1\}$.

The points of $\mathcal{A} \cap \ell$ are the $\frac{q}{4}$ points of $\ell \setminus \{N\}$ that are not contained in $\mathcal{D}_{12} \cup \mathcal{D}_{13} \cup \mathcal{D}_{14}$. Recall that $\mathcal{D}_{12} = \{(x, 1, 0) \mid \text{Tr}(x) = 1\}$, $\mathcal{D}_{13} = \{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(\frac{x}{\alpha}) = a + 1\}$ and $\mathcal{D}_{14} = \{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) + \text{Tr}(\frac{x}{\alpha}) = a + 1\}$. So the points of $\mathcal{A} \cap \ell$ are the points $\{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\frac{x}{\alpha}) = a\}$.

Note that the parameters ξ, γ, δ and ε are written in function of α and β in the following way:

$$\gamma = \frac{\beta + 1}{\alpha\beta + 1}, \quad \delta = \beta + 1, \quad \varepsilon = \frac{(\alpha + 1)\beta}{\alpha\beta + 1}, \quad \xi = \alpha\beta\gamma.$$

The conditions on these parameters that we found during the proof, namely

$$\alpha, \xi, \beta, \gamma, \delta, \varepsilon \neq 0 \quad \alpha\beta, \alpha\gamma \neq 1 \quad \varepsilon \neq (\alpha + 1)\gamma, \delta \neq (\alpha + 1)\beta \quad \beta \neq \gamma, \xi \neq 1,$$

can thus be rewritten as conditions on α and β . We find that $\alpha, \beta \notin \{0, 1\}$, $\alpha\beta \neq 1$ and $\alpha\beta^2 \neq 1$. So, indeed \mathcal{A} is PGL-equivalent to a KM-arc given in Theorem 4.6. In fact, all examples in Theorem 4.6 are described except the ones with $\alpha\beta^2 = 1$. This finishes the proof. \square

Theorem 4.13. *Let \mathcal{A} be a KM-arc of type $\frac{q}{4}$ in $\text{PG}(2, q)$, with $q = 2^h$. If ℓ is a $\frac{q}{4}$ -secant of \mathcal{A} such that \mathcal{A} has property (I) with respect to ℓ , then \mathcal{A} is PGL-equivalent to an arc given in Theorem 4.6 with $\alpha\beta^2 = 1$, and hence, is a translation arc with translation line ℓ .*

Proof. Recall that Tr is the absolute trace function from \mathbb{F}_q to \mathbb{F}_2 . Let N be the nucleus of \mathcal{A} . Denote the four $\frac{q}{4}$ -secants of \mathcal{A} different from ℓ by $\ell_1, \ell_2, \ell_3, \ell_4$, and denote $\mathcal{A} \cap \ell_i$ by \mathcal{A}_i and $\mathcal{A}_i \cup \{N\}$ by \mathcal{A}_i^N , $i = 1, \dots, 4$. We also use the notations

$$\mathcal{D}_{ij} = \{\langle P, Q \rangle \cap \ell \mid P \in \mathcal{A}_i, Q \in \mathcal{A}_j\} \text{ and } \mathcal{D}_{ij}^N = \mathcal{D}_{ij} \cup \{N\} \quad 1 \leq i < j \leq 4.$$

Since \mathcal{A} has property (I) we know that $\mathcal{D}_{12} = \mathcal{D}_{34}$ and that \mathcal{D}_{12}^N is an $(h-2)$ -club of rank $h-1$ with head N in ℓ . Then we can find two $(h-1)$ -clubs of rank h with head N , say \mathcal{C} and \mathcal{C}' , such that $\mathcal{D}_{12}^N = \mathcal{C} \cap \mathcal{C}'$. By Theorem 2.3 we can choose a frame of $\text{PG}(2, q)$ such that \mathcal{C} is the set $\{(x, \text{Tr}(x), 0) \mid x \in \mathbb{F}_q^*\}$ and such that $(0, 0, 1) \in \mathcal{A}_1$. It follows that $N = (1, 0, 0)$. By Theorem 2.3 all $(h-1)$ -clubs of rank h are equivalent, so \mathcal{C}' is given by $\{(\frac{x}{\beta} + \beta' \text{Tr}(x), \text{Tr}(x), 0) \mid x \in \mathbb{F}_q^*\}$ for some $\beta \in \mathbb{F}_q \setminus \{0, 1\}$ and some $\beta' \in \mathbb{F}_q$. Denote $\text{Tr}(\beta\beta') = b$. Hence, \mathcal{D}_{12} is given by $\{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\}$.

Now, let \mathcal{D} be a Desarguesian spread in $\text{PG}(3h-1, 2)$, let H be the $(2h-1)$ -space such that $\ell = \mathcal{B}(H)$, and let $\rho \in \mathcal{D}$ be the spread element such that $\mathcal{B}(\rho) = N$.

We define σ as the $(2h)$ -space in $\text{PG}(3h-1, 2)$ spanned by H and $(0, 0, 1)_{\mathbb{F}_2}$. Then there is a unique $(h-2)$ -space $\sigma_i \subset \sigma$ such that $\mathcal{B}(\sigma_i) = \mathcal{A}_i^N$, $i = 1, \dots, 4$. For every $i = 1, \dots, 4$, the subspace σ_i meets ρ in an $(h-3)$ -space; the existence of these subspaces σ_i follows from the first condition for KM-arcs of property (I). Let μ_{ij} be the subspace $H \cap \langle \sigma_i, \sigma_j \rangle$, for $1 \leq i < j \leq 4$. Then, $\mathcal{D}_{ij}^N = \mathcal{B}(\mu_{ij})$, for $1 \leq i < j \leq 4$ and $\mu_{ij} \subset H$ is an $(h-2)$ -space meeting ρ in an $(h-3)$ space, $1 \leq i < j \leq 4$. However, $\mu_{ij} \cap \rho$ must contain the $(h-3)$ -spaces $\sigma_i \cap \rho$ and $\sigma_j \cap \rho$, so necessarily $\mu_{ij} \cap \rho = \sigma_i \cap \rho = \sigma_j \cap \rho$. We find that there is an $(h-3)$ -space $\tau \subset \rho$ such that $\sigma_i \cap \rho = \tau$ for $i = 1, \dots, 4$, and such that $\tau = \mu_{ij} \cap \rho$ for $1 \leq i < j \leq 4$.

Let μ and μ' be the two $(h-1)$ -spaces in H such that $\mu_{12} = \mu \cap \mu'$ and such that $\mathcal{B}(\mu) = \mathcal{C}$ and $\mathcal{B}(\mu') = \mathcal{C}'$. Now, after having fixed the spread \mathcal{D} , we still can choose coordinates for H in such a way that $\mu = \{(x, \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*\}$. Then, the space μ' is given by $\{(\frac{x}{\beta} + \beta' \text{Tr}(x), \text{Tr}(x), 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*\}$. Hence, the points of $\mu_{12} \setminus \tau$ are given by $\{(x, 1, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\}$, and τ is given by $\{(x, 0, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q^*, \text{Tr}(x) = \text{Tr}(\beta x) = 0\}$.

The $(h-2)$ -space σ_1 is the space $\langle \tau, (0, 0, 1)_{\mathbb{F}_2} \rangle$, and hence the points of \mathcal{A}_1 are given by $\{(x, 0, 1) \mid \text{Tr}(x) = \text{Tr}(\beta x) = 0\}$. The subspace σ_2 is the unique $(h-2)$ -space through τ in the $(h-1)$ -space $\langle \mu_{12}, \sigma_1 \rangle$, different from μ_{12} and σ_1 . It follows immediately that $\mathcal{A}_2 = \{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\}$.

For all $(h-2)$ -spaces in H meeting ρ in the $(h-3)$ -space τ the points of this space outside τ are given by $\{(x, \zeta, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = e, \text{Tr}(\beta x) = e'\}$ for a $\zeta \in \mathbb{F}_q^*$ and $e, e' \in \mathbb{F}_2$. So, there exist $\eta, \theta \in \mathbb{F}_q^*$ and $d_3, d'_3, d_4, d'_4 \in \mathbb{F}_2$ such that $\mu_{13} = \{(x, \eta, 0)_{\mathbb{F}_2} \mid x \in$

$\mathbb{F}_q, \text{Tr}(x) = d_3, \text{Tr}(\beta x) = d'_3\}$ and $\mu_{14} = \{(x, \theta, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_4, \text{Tr}(\beta x) = d'_4\}$. It follows that the points of $\sigma_3 \setminus \tau$ are the points $\{(x, \eta, 1)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3, \text{Tr}(\beta x) = d'_3\}$ and thus $\mathcal{A}_3 = \{(x, \eta, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3, \text{Tr}(\beta x) = d'_3\}$. Analogously, $\sigma_4 \setminus \tau = \{(x, \theta, 1)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_4, \text{Tr}(\beta x) = d'_4\}$ and $\mathcal{A}_4 = \{(x, \theta, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_4, \text{Tr}(\beta x) = d'_4\}$. Necessarily, $1 \neq \eta \neq \theta \neq 1$.

We consider a line passing through an arbitrary point of \mathcal{A}_1 and an arbitrary point of \mathcal{A}_2 , the line $\ell_{12} : \langle (x, 0, 1), (y, 1, 1) \rangle$ with $\text{Tr}(x) = 0 = \text{Tr}(\beta x)$ and $\text{Tr}(y) = 1, \text{Tr}(\beta y) = b$. This line cannot contain a point of \mathcal{A}_3 since \mathcal{A} is a KM-arc. Hence, the point $((\eta + 1)x + \eta y, \eta, 1) \in \ell_{12}$ is not a point of \mathcal{A}_3 . So, $\text{Tr}(\eta(x + y)) \neq d_3$ or $\text{Tr}(\eta\beta(x + y)) \neq d'_3$. Analogously, considering \mathcal{A}_4 we find that $\text{Tr}(\theta(x + y)) \neq d_4$ or $\text{Tr}(\theta\beta(x + y)) \neq d'_4$. Consequently, in general the systems of equations

$$\begin{cases} \text{Tr}(z) = 1 \\ \text{Tr}(\beta z) = b \\ \text{Tr}(\eta z) = d_3 \\ \text{Tr}(\eta\beta z) = d'_3 \end{cases} \quad \text{and} \quad \begin{cases} \text{Tr}(z) = 1 \\ \text{Tr}(\beta z) = b \\ \text{Tr}(\theta z) = d_4 \\ \text{Tr}(\theta\beta z) = d'_4 \end{cases} \quad (1)$$

cannot have a solution. We express that this systems of equations cannot have a solution, using Lemma 4.5. We find that $\eta, \theta \in \{\beta, \beta + 1, \frac{1}{\beta}, \frac{1}{\beta} + 1, \frac{1}{\beta+1}, \frac{\beta}{\beta+1}\}$ and for each of these choices for η or θ there is a corresponding relation on d_3 or d'_3 , or on d_4 or d'_4 , respectively.

Claim: we claim that two different $(h - 2)$ -spaces in H meeting ρ in τ cannot give rise to the same linear set in $\mathcal{B}(H)$ unless $\beta^2 + \beta + 1 = 0$. We look at two arbitrary $(h - 2)$ -spaces ν_1 and ν_2 in H meeting ρ in τ , whose points outside τ are given by $\{(x, \zeta_1, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = e_1, \text{Tr}(\beta x) = e'_1\}$ and $\{(x, \zeta_2, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = e_2, \text{Tr}(\beta x) = e'_2\}$, respectively. The sets $\mathcal{B}(\nu_1) \setminus \{N\}$ and $\mathcal{B}(\nu_2) \setminus \{N\}$ are given by $\{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(\zeta_1 x) = e_1, \text{Tr}(\zeta_1 \beta x) = e'_1\}$ and $\{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(\zeta_2 x) = e_2, \text{Tr}(\zeta_2 \beta x) = e'_2\}$, respectively. By Lemma 4.5 the sets $\mathcal{B}(\nu_1)$ and $\mathcal{B}(\nu_2)$ are equal if and only if either $(\zeta_1, e_1, e'_1) = (\zeta_2, e_2, e'_2)$ or else $\beta^2 + \beta + 1 = 0$ and $(\zeta_1, e_1, e'_1) \in \{(\beta\zeta_2, e'_2, e_2 + e'_2), ((\beta + 1)\zeta_2, e_2 + e'_2, e_2)\}$. In the first case we find $\nu_1 = \nu_2$. We conclude that if different ν_1 and ν_2 determine the same linear set on ℓ then $\beta^2 + \beta + 1 = 0$. We note that such an element β only can exist if \mathbb{F}_4 is a subfield of \mathbb{F}_q , equivalently if h is even. This finishes the proof of the claim.

We consider the $(h - 2)$ -space $\mu_{34} = \langle \sigma_3, \sigma_4 \rangle \cap H$, which is given by $\tau \cup \{(x, \eta + \theta, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3 + d_4, \text{Tr}(\beta x) = d'_3 + d'_4\}$. Now, we distinguish between two cases. First, we assume that $\mu_{12} \neq \mu_{34}$. On the one hand, since $\mathcal{B}(\mu_{12}) = \mathcal{B}(\mu_{34})$, we must have by our claim that $\beta^2 + \beta + 1 = 0$ and $\eta + \theta \in \{\beta, \beta + 1\}$ as $\mu_{12} \setminus \tau$ is given by $\{(x, 1, 0)_{\mathbb{F}_2} \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\}$. On the other hand $\eta, \theta \in \{\beta, \beta + 1, \frac{1}{\beta}, \frac{1}{\beta} + 1, \frac{1}{\beta+1}, \frac{\beta}{\beta+1}\} = \{\beta, \beta + 1\}$ as solutions of the system of equations in (1). These two results contradict each other, since it follows from $\eta, \theta \in \{\beta, \beta + 1\}$ that $\eta + \theta \in \{0, 1\}$.

So, secondly we assume that $\mu_{12} = \mu_{34}$ and hence $\theta = \eta + 1$ and $(d_4, d'_4) = (d_3 + 1, d'_3 + b)$. We expressed before that the systems of equations in (1) do not have a solution, and so we find that $\{\eta, \theta\}$ equals $\{\beta, \beta + 1\}$, $\{\frac{1}{\beta}, \frac{1}{\beta} + 1\}$ or $\{\frac{1}{\beta+1}, \frac{\beta}{\beta+1}\}$, and for each of these solutions there is a corresponding relation on d_3 or d'_3 . As η and θ are interchangeable we can choose $\eta \in \{\beta, \frac{1}{\beta}, \frac{1}{\beta+1}\}$. We distinguish between these three cases.

- If $\eta = \beta$, then expressing that the system of equations (1) does not have a solution and using Lemma 4.5, it follows that $b = d_3 + 1$. We define $a = d'_3 + 1$ and we find,

using $(d_4, d'_4) = (d_3 + 1, d'_3 + b)$

$$\begin{aligned}\mathcal{A}_1 &= \{(x, 0, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta x) = 0\} \\ \mathcal{A}_2 &= \{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\} \\ \mathcal{A}_3 &= \{(x, \beta, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b + 1, \text{Tr}(\beta x) = a + 1\} \\ \mathcal{A}_4 &= \{(x, \beta + 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b, \text{Tr}(\beta x) = a + b + 1\} .\end{aligned}\quad (2)$$

- If $\eta = \frac{1}{\beta}$, then from Lemma 4.5 it follows that $d'_3 = 0$. Again using $(d_4, d'_4) = (d_3 + 1, d'_3 + b)$, we find

$$\begin{aligned}\mathcal{A}_1 &= \{(x, 0, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta x) = 0\} \\ \mathcal{A}_2 &= \{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\} \\ \mathcal{A}_3 &= \{(x, \frac{1}{\beta}, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3, \text{Tr}(\beta x) = 0\} \\ \mathcal{A}_4 &= \{(x, \frac{1}{\beta} + 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3 + 1, \text{Tr}(\beta x) = b\} .\end{aligned}$$

Using the transformation matrices $\begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $(\kappa_1, \kappa_2) = (1, \beta)$ if $d_3 = 1$, with $(\kappa_1, \kappa_2) = (\beta, 1)$ if $b = 1$, and with $(\kappa_1, \kappa_2) = (\beta + 1, \frac{\beta}{\beta+1})$ if $b = d_3$, and setting $(\beta', a', b') = (\beta, b + 1, 0)$, $(\beta', a', b') = (\frac{1}{\beta}, d_3 + 1, 1)$ and $(\beta', a', b') = (\frac{1}{\beta+1}, b + 1, b + 1)$ respectively, we find that the sets $\mathcal{A}_i, i = 1, \dots, 4$, are PGL-equivalent to

$$\begin{aligned}\mathcal{A}'_1 &= \{(x, 0, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta' x) = 0\} \\ \mathcal{A}'_2 &= \{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta' x) = b'\} \\ \mathcal{A}'_3 &= \{(x, \beta', 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b' + 1, \text{Tr}(\beta' x) = a' + 1\} \\ \mathcal{A}'_4 &= \{(x, \beta' + 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b', \text{Tr}(\beta' x) = a' + b' + 1\} .\end{aligned}$$

We find the same sets as in (2) with the parameters (a, b) replaced by (a', b') .

- If $\eta = \frac{1}{\beta+1}$, then from Lemma 4.5 it follows that $d'_3 = d_3$. Again using $(d_4, d'_4) = (d_3 + 1, d'_3 + b)$, we find

$$\begin{aligned}\mathcal{A}_1 &= \{(x, 0, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta x) = 0\} \\ \mathcal{A}_2 &= \{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\} \\ \mathcal{A}_3 &= \{(x, \frac{1}{\beta+1}, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3, \text{Tr}(\beta x) = d_3\} \\ \mathcal{A}_4 &= \{(x, \frac{\beta}{\beta+1}, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = d_3 + 1, \text{Tr}(\beta x) = d_3 + b\} .\end{aligned}$$

Using the transformation matrices $\begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with $(\kappa_1, \kappa_2) = (1, \beta + 1)$ if $d_3 = 1$, with $(\kappa_1, \kappa_2) = (\beta, 1 + \frac{1}{\beta})$ if $d_3 = b + 1$, and with $(\kappa_1, \kappa_2) = (\beta + 1, 1)$ if $b = 0$, and setting $(\beta', a', b') = (\beta, b, 1)$, $(\beta', a', b') = (\frac{1}{\beta}, b, b)$ and $(\beta', a', b') = (\frac{1}{\beta+1}, d_3 + 1, 1)$

respectively, we find that these sets are PGL-equivalent to

$$\begin{aligned}\mathcal{A}'_1 &= \{(x, 0, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta'x) = 0\} \\ \mathcal{A}'_2 &= \{(x, 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta'x) = b'\} \\ \mathcal{A}'_3 &= \{(x, \beta', 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b' + 1, \text{Tr}(\beta'x) = a' + 1\} \\ \mathcal{A}'_4 &= \{(x, \beta' + 1, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b', \text{Tr}(\beta'x) = a' + b' + 1\}.\end{aligned}$$

We find the same sets as in (2) with the parameters (a, b) replaced by (a', b') .

So, in all three cases we may assume $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and \mathcal{A}_4 are given by descriptions given in (2). Now, the sets \mathcal{D}_{ij} are given by:

$$\begin{aligned}\mathcal{D}_{12} &= \mathcal{D}_{34} = \{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1, \text{Tr}(\beta x) = b\} \\ \mathcal{D}_{13} &= \mathcal{D}_{24} = \{(x, \beta, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b + 1, \text{Tr}(\beta x) = a + 1\} \\ &= \{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(\beta x) = b + 1, \text{Tr}(\beta^2 x) = a + 1\} \\ \mathcal{D}_{14} &= \mathcal{D}_{23} = \{(x, \beta + 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = b, \text{Tr}(\beta x) = a + b + 1\} \\ &= \{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}((\beta + 1)x) = b, \text{Tr}((\beta^2 + \beta)x) = a + b + 1\}\end{aligned}$$

The set $\mathcal{A} \cap \ell$ is the set of points on $\ell \setminus \{N\}$, not in $\mathcal{D}_{12} \cap \mathcal{D}_{13} \cap \mathcal{D}_{14}$. By the previous $\mathcal{A} \cap \ell$ is the set $\{(x, 1, 0) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta^2 x) = a\}$. So, \mathcal{A} is PGL-equivalent to a KM-arc given in Theorem 4.6, namely with $\alpha = \frac{1}{\beta^2}$, and consequently $\gamma = \beta$. This finishes the proof in case \mathcal{A} has property (I). \square

By combining Lemma 4.2, Lemma 3.10 and Theorem 4.13 we obtain the following corollary.

Corollary 4.14. *A KM-arc \mathcal{A} of type $q/4$ is a translation arc with translation line ℓ if and only if \mathcal{A} satisfies Property (I) with respect to ℓ . All translation KM-arcs in $\text{PG}(2, q)$ of type $\frac{q}{4}$, including the example of Vandendriessche, can be obtained from Theorem 4.6.*

We now exploit the link between $(h - 2)$ -clubs of rank h in $\text{PG}(1, q)$, $q = 2^h$, and translation KM-arcs of type $q/4$ to obtain a classification of $(h - 2)$ -clubs of rank h .

Corollary 4.15. *Let \mathcal{C} be an $(h - 2)$ -club of rank h with head N contained in the line $\ell = \text{PG}(1, q)$, $q = 2^h$. Then the set $\ell \setminus \mathcal{C}$ together with the head N is an $(h - 2)$ -club of rank $h - 1$. Moreover, \mathcal{C} is equivalent to the set of points $(1, 0) \cup \{(x, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1 \vee \text{Tr}(\varepsilon x) = c\}$ for some $\varepsilon \in \mathbb{F}_q$ and $c \in \mathbb{F}_2$.*

Proof. If \mathcal{C} is an $(h - 2)$ -club of rank h in $\text{PG}(1, q)$, $q = 2^h$, then by Theorem 2.1, \mathcal{C} defines a translation KM-arc \mathcal{A} of type $q/4$ with translation line ℓ . By Theorem 4.13, \mathcal{A} is PGL-equivalent to a KM-arc obtained by the construction of Theorem 4.6 with $\alpha = \frac{1}{\beta^2}$. This implies that $\mathcal{A} \cap \ell = \{(x, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 0, \text{Tr}(\beta^2 x) = a\}$ for some $\beta \in \mathbb{F}_{2^h}^*$ and $a \in \mathbb{F}_2$, which forms together with the point N with coordinates $(1, 0)$ an $(h - 2)$ -club of rank $h - 1$.

The set $\mathcal{A} \cap \ell$ is precisely the complement of the set \mathcal{C} . Hence, we can describe \mathcal{C} as the set of points $(1, 0) \cup \{(x, 1) \mid x \in \mathbb{F}_q, \text{Tr}(x) = 1 \vee \text{Tr}(\varepsilon x) = c\}$, for some $\varepsilon \in \mathbb{F}_{2^h}$ and $c \in \mathbb{F}_2$, where we have used that every element of \mathbb{F}_{2^h} is a square. \square

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